

# Scalability of Greenberger-Horne-Zeilinger and random-state entanglement in the presence of decoherence

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We derive analytical upper bounds for the entanglement of generalized Greenberger-Horne-Zeilinger states coupled to locally depolarizing and dephasing environments and for local thermal baths of arbitrary temperature. These bounds apply for *any* convex quantifier of entanglement, and exponential entanglement decay with the number of constituent particles is found. The bounds are tight for depolarizing and dephasing channels. We also show that randomly generated initial states tend to violate these bounds and that this discrepancy grows with the number of particles.

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## I. INTRODUCTION

Impressive experimental progress in the manipulation of composite quantum systems toward real-world applications of quantum information and communication theory has taken place over the last few years. The controlled production of genuinely multipartite entangled states plays a central role in this program and is crucial for the scaling of existing toy quantum protocols to mature technologies. Among those, the  $W$  [1] and Greenberger-Horne-Zeilinger (GHZ) [2] types of entanglement play a paradigmatic role since they incarnate characteristic traits and subtleties of multipartite entanglement which allow, e.g., to implement protocols for secure quantum communication.

Three photon  $W$ -type entanglement [3,4] and five- and six-photon GHZ entangled states [5,6] have already been observed. Very recently, a ten-party GHZ state was produced using five hyperentangled photons [7]. GHZ states have also been reported in cavity QED experiments [8] and for three [9], four [10], and six [11] trapped ions. Eight ions were prepared in a  $W$  state [12], and the controlled generation of different multipartite entanglement families was shown in [13]. Furthermore, the recent implementation of a robust and extremely high-fidelity entangling gate of two ions [14] opens the way to the controlled production of GHZ states of a few tens of ions.

However, it is known that scaling multipartite entangled states up to many constituents is haunted by the decoherence processes arising from the unavoidable and (most times) detrimental interaction of the system degrees of freedom with the environment. Complete disentanglement may even occur at finite times [15–22,26–34], as experimentally demonstrated in [23–25]. It is therefore vital for experimental implementations to predict the time scales on which appreciable amounts of entanglement will prevail under realistic assumptions on the environment coupling. Theoretical studies of disentanglement dynamics for large systems were presented in [16,18,26–34], in particular for GHZ and  $W$  type states. While Refs. [16,18,29] considered the scaling of the

disentanglement time with the system size, Refs. [31,26,32] put the focus on the scaling of the short time scale behavior of entanglement, which allows for a much better characterization of the robustness of the initial state's entanglement under noise. In particular, it was shown in [31] that even if the disentanglement time of GHZ states grows with the number of parties  $N$ , the residual entanglement (there quantified by the negativity [35]) is reduced to arbitrarily small values at times which decrease with  $N$ .

In the present paper, we expand the above studies to larger classes of multipartite entangled states. We first investigate the scaling properties of disentanglement of mixed-state generalizations of GHZ states, namely, generalized GHZ-diagonal states. For systems subject to natural decoherence models, analytical upper bounds can be derived for their entanglement and for the associated scaling behavior with the system size. Our findings are valid for *any* convex entanglement measure. In addition, we numerically study the entanglement dynamics of randomly generated states, quantified in terms of the respective states' negativity. Random samples of pure initial states do not abide to the above-mentioned bounds. Our numerical data suggest that the latter discrepancy increases with the number of system constituents, which gives evidence of the exponential fragility of GHZ entanglement against decoherence *not* being a generic feature.

The paper is organized as follows. In Sec. II, we introduce our notation and define the generalizations of the GHZ states to be studied later. Section III defines our environment models. The analytical upper bounds for the entanglement, together with the resulting scaling behavior, are derived in Sec. IV, where the tightness of these bounds is also assessed. Section V compares the scaling of the properties of GHZ entanglement to those of random pure states. Finally, Sec. VI summarizes our conclusions.

## II. GENERALIZED GHZ AND GHZ-DIAGONAL STATES

Originally, the GHZ state [2] was defined for three qubits as a superposition of the many particle states of all of them

being in their respective basis state  $|0\rangle$  and all of them being in the orthogonal state  $|1\rangle$ . This notion straightforwardly extends to more qubits and gives the GHZ state to be

$$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000 \dots 0\rangle + |111 \dots 1\rangle). \quad (1)$$

Without imposing more details on the nature of the specific quantum system the basis states  $|0\rangle$  and  $|1\rangle$  are abstract and without any physical meaning. Thus, formally, the state  $(|010\rangle + |101\rangle)/\sqrt{2}$  for three qubits, e.g., follows the spirit of the GHZ construction in precisely the same manner. When considering all such states it is helpful to interpret a string of zeros and ones as the binary representation of a number; the number of the second term always being bit-wise inverted with respect to the first. A many particle basis state of  $N$  qubits can thus be labeled with numbers from 0 (all zeros) to  $2^N - 1$  (all ones). Our three qubit example would therewith read  $(|2\rangle + |5\rangle)/\sqrt{2}$ . If we furthermore allow for different amplitudes of the vectors and, for later convenience, treat a phase difference of  $\pi$  separately (thus distinguishing states of even and odd parity), we arrive at the *generalized GHZ states* of  $N$  qubits

$$|\psi_k^\pm(\alpha, \beta)\rangle \equiv \alpha|k\rangle \pm \beta|\bar{k}\rangle, \quad (2)$$

with an  $N$ -bit number  $k$ ,  $0 \leq k \leq 2^N - 1$ ,  $\bar{k}$  the bit-wise flipped number, and complex amplitudes  $\alpha$  and  $\beta$  such that  $|\alpha|^2 + |\beta|^2 = 1$  and  $\alpha, \beta \neq 0$ .

An incoherent mixture of several generalized GHZ states  $|\psi_k^\pm(\alpha, \beta)\rangle$  with different  $k$  and parity, but the same amplitudes  $\alpha$  and  $\beta$ , is a *generalized GHZ-diagonal state* [36]

$$\rho = \sum_{k=0}^{2^N-1} (\lambda_k^+ |\psi_k^+(\alpha, \beta)\rangle \langle \psi_k^+(\alpha, \beta)| + \lambda_k^- |\psi_k^-(\alpha, \beta)\rangle \langle \psi_k^-(\alpha, \beta)|). \quad (3)$$

The coefficients  $\lambda_k^\pm$  denote the respective probabilities with which the state appears in the mixture. Naturally, they are positive and sum up to 1.

### III. NOISE MODELS

Given the above class of initial states, we proceed now to describe three paradigmatic models of incoherent dynamics present in typical experimental settings. We assume that qubits do not interact with each other, neither directly nor indirectly, through their baths, i.e., they feel independent and local environments. This is a reasonable approach for qubits being located in different laboratories or sufficiently well isolated from each other. Furthermore, we assume that the qubit-bath interaction is identical for all of them.

We describe the state evolution by means of a map (or channel) such that a single qubit's initial state  $\rho_i$  is mapped

onto its final state by virtue of  $\rho_f = \mathcal{E}(\rho_i)$ . Since the mapping is between quantum states, it needs to be completely positive and trace preserving. Every such map allows for an operator sum (or Kraus) representation [37]

$$\mathcal{E}(\rho_i) = \sum_j K_j \rho_i K_j^\dagger, \quad (4)$$

with Kraus operators  $K_j$  fulfilling  $\sum_j K_j^\dagger K_j = 1$ . The initial joint state  $\rho$  of  $N$  qubits evolves thus according to the  $N$ -fold tensor product of the individual maps

$$\rho \equiv \Lambda(\rho) \equiv \underbrace{\mathcal{E} \otimes \mathcal{E} \otimes \dots \otimes \mathcal{E}}_N(\rho). \quad (5)$$

Specifically, the channels we consider [37] are the depolarizing channel (that occurs, e.g., in spin scattering), the dephasing (or phase damping) channel (typical of elastic collisional interactions), and a thermal bath at arbitrary temperature (provided, e.g., by a thermal radiation background). The depolarizing channel (for which we will use the subscript  $D$  in the sequel) describes the situation in which the environment isotropically destroys the information on a qubit's state and thus steers it into a maximally mixed state  $\mathbb{1}/2$ . The characteristic quantity describing the dynamics is the probability  $p$  of finding the state completely depolarized. The single qubit's initial state  $\rho_i$  thus evolves to  $\mathcal{E}_D(\rho_i) = \rho_i(1-p) + \mathbb{1}/2$ . The corresponding Kraus representation allows for a convenient form in terms of the Pauli matrices

$$\mathcal{E}_D(\rho_i) = \sum_{j=0}^3 s_j \sigma_j \rho_i \sigma_j, \quad (6)$$

where  $s_0 \equiv 1 - 3p/4$ ,  $s_1 = s_2 = s_3 \equiv p/4$ ,  $\sigma_0 \equiv \mathbb{1}$ , and  $\sigma_1, \sigma_2$ , and  $\sigma_3$  are the three familiar Pauli operators associated with the respective qubit.

As a second type of noise, we discuss the phase damping channel (subscript PD). It represents a situation when quantum coherence is lost *without* any population or excitation exchange. If the probability of complete phase loss is  $p$ , its Kraus representation reads

$$\mathcal{E}_{\text{PD}}(\rho_i) = (1-p)\rho_i + p(|0\rangle\langle 0| \rho_i |0\rangle\langle 0| + |1\rangle\langle 1| \rho_i |1\rangle\langle 1|), \quad (7)$$

with Kraus operators

$$M_0 = \sqrt{1-p}, \quad M_1 = |0\rangle\langle 0| \sqrt{p}, \quad M_2 = |1\rangle\langle 1| \sqrt{p}. \quad (8)$$

Finally, the third type of environment to be dealt with here is a thermal bath (subscript  $T$ ). In this scenario, the qubit's basis states  $|0\rangle$  and  $|1\rangle$  function as ground and excited states, respectively, in order to exchange excitations with the bath. In the Born–Markov approximation [38,39] this yields the Kraus operators

$$\begin{aligned}
K_0 &= \sqrt{\frac{\bar{n}+1}{2\bar{n}+1}}(|0\rangle\langle 0| + |1\rangle\langle 1|\sqrt{1-p}), \\
K_1 &= \sqrt{\frac{\bar{n}+1}{2\bar{n}+1}}|0\rangle\langle 1|\sqrt{p}, \\
K_2 &= \sqrt{\frac{\bar{n}}{2\bar{n}+1}}(|0\rangle\langle 0|\sqrt{1-p} + |1\rangle\langle 1|), \\
\text{and } K_3 &= \sqrt{\frac{\bar{n}}{2\bar{n}+1}}|1\rangle\langle 0|\sqrt{p}, \tag{9}
\end{aligned}$$

where the average excitation  $\bar{n}$  of the bath modes induces an energy exchange from the bath to the system and vice versa. For zero temperature, i.e.,  $\bar{n}=0$ , this reduces to the amplitude damping channel, where system excitations irreversibly dissipate into the bath, triggered by the vacuum fluctuations of the bath mode, with a rate  $\gamma$ . The opposite case, at infinite temperature, is established by taking the limit  $\bar{n}\rightarrow\infty$  and  $\gamma\rightarrow 0$ , with  $\bar{n}\gamma=\Gamma\equiv\text{const}$ , and models a purely diffusive environment with diffusion constant  $\Gamma$ . In this case,  $p$  is the probability for an excitation exchange between system and bath.

We stress that in all three cases, the models presented can encompass many different dynamics, depending on how one relates  $p$  to time  $t$ . For instance, the thermal bath at zero temperature can model an atom interacting with the free electromagnetic field if we take  $p=1-\exp(-\gamma t/2)$  or with the field in a cavity by taking  $p=\sin^2(\omega t/2)$ , in which Rabi oscillations (of vacuum Rabi frequency  $\omega$ ) can take place.

#### IV. BOUNDS ON THE ROBUSTNESS OF ENTANGLEMENT

With the detailed description of the states and the dynamics to be scrutinized at hand we can assess the robustness of the system's entanglement by bounding it from above as it evolves according to such open dynamics. We only assume two basic properties of the entanglement quantifier  $E$  and thus provide results for *any* such entanglement quantifier. We merely require  $E$  to vanish for separable states and not to increase when probabilistically mixing two states  $\sigma$  and  $\omega$ , i.e., it needs to be convex

$$E[\mu\sigma + (1-\mu)\omega] \leq \mu E(\sigma) + (1-\mu)E(\omega) \tag{10}$$

for any  $\mu \in [0, 1]$ .

The basic idea behind the derivation of the upper bounds is to decompose all studied states as convex combinations of an entangled and a separable part, i.e.,  $\rho = \mu_{\text{ent}}\rho_{\text{ent}} + (1-\mu_{\text{ent}})\rho_{\text{sep}}$ , and to use the properties listed just above to bound the entanglement evolution as  $E(\rho) \leq \mu_{\text{ent}}E(\rho_{\text{ent}})$ . In what follows we identify such decompositions for initial states of the general GHZ or general GHZ-diagonal type under the influence of the paradigmatic environments considered in Sec. III.

#### A. Depolarization

We begin with the case of open system dynamics that independently depolarizes each qubit. For this type of decoherence process, we prove now the following:

(i) The entanglement of an initially generalized GHZ-diagonal state  $\rho$  subject to local depolarizing channel  $\Lambda_D$  is bounded from above as

$$E(\Lambda_D(\rho)) \leq (1-p)^N E(\rho).$$

To prove this statement, let us first begin by considering a single generalized GHZ state, namely, the particular case  $\rho_0 \equiv |\psi_0^+(\alpha, \beta)\rangle\langle\psi_0^+(\alpha, \beta)|$ .

The evolved density matrix  $\Lambda_D(\rho_0)$  is straightforwardly obtained by the  $N$ -fold application of the depolarizing channel (6) of one qubit according to Eq. (5). In the resulting  $N$ -fold operator sum we first focus on one of the sums for a single qubit. There the two terms for  $\sigma_1$  and  $\sigma_2$  yield contributions of equal weight ( $s_1=s_2$ ) with different signs in their coherences because  $\sigma_1$  does a bit flip whereas  $\sigma_2$  results in a combined bit and phase flip (parity change). Altogether, this cancels the coherences and thus results in a diagonal and hence separable contribution. Thus, again considering all the sums, the only terms not immediately causing separable contributions are the applications of only the identity ( $\sigma_0$ ) or the parity changing operator  $\sigma_3$ .

Therefore, after the application of channel  $\Lambda_D$ , we identify three contributions to the final state. First, there is the unchanged state  $|\psi_0^+(\alpha, \beta)\rangle\langle\psi_0^+(\alpha, \beta)|$  resulting from the application of only identities or an even number of parity changes. Second, there is the parity-changed counterpart  $|\psi_0^-(\alpha, \beta)\rangle\langle\psi_0^-(\alpha, \beta)|$  that is generated by an odd number of parity changes. Lastly, there is a separable contribution originating from the application of at least one of  $\sigma_1$  or  $\sigma_2$  to one of the qubits

$$\begin{aligned}
\Lambda_D(\rho_0) &= \lambda_+ |\psi_0^+(\alpha, \beta)\rangle\langle\psi_0^+(\alpha, \beta)| + \lambda_- |\psi_0^-(\alpha, \beta)\rangle\langle\psi_0^-(\alpha, \beta)| \\
&\quad + \lambda_{\text{sep}} \rho_{\text{sep}}. \tag{11}
\end{aligned}$$

Here, the first two terms are of opposite parities. Their coherences partially cancel when being summed, yielding another diagonal (separable) contribution. Their difference is what determines the remaining entangled contribution in the decomposition. Its careful evaluation [see Appendix] yields

$$\Lambda_D(\rho_0) = (1-p)^N \rho_0 + [1 - (1-p)^N] \rho'_{\text{sep}}. \tag{12}$$

This result does not depend on the fact that it was derived using  $|\psi_k^+(\alpha, \beta)\rangle$  with  $k=0$  as initial state and holds for different  $k$  as well. Also, since the application of channels is linear this result extends immediately to any convex combination such as the generalized GHZ-diagonal states given in Eq. (3). The convexity property (10) then yields the desired result.

As a corollary of (i), we have:

(ii) For the special case of two qubits ( $N=2$ ), the bound in (i) holds for *all* initial two-qubit states (not only the general GHZ-diagonal ones).

Consider first the pure-state case. Any pure two-qubit state  $|\Psi\rangle$  can be expressed in local product-state bases such that it is  $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle \equiv |\psi_0^+(\alpha, \beta)\rangle$ , with real and posi-

tive  $\alpha$  and  $\beta$  [37]. Then, for *any* pure two-qubit state, there is always a local basis (the Schmidt basis) in which it is a generalized GHZ state. Now, since  $\mathcal{E}_D$  is basis independent—it shrinks the Bloch sphere without distinction of any particular direction—the local basis to which all four Kraus operators in (6) make reference can be any, in particular the one in which  $|\Psi\rangle$  is a generalized GHZ state. By employing (i), we have that (ii) holds for *all* pure two-qubit states.

For the mixed-state case, we consider *any* two-qubit state  $\rho$  in its *convex roof* [40] decomposition  $\rho \equiv \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$  (for which the entanglement and the average entanglement coincide) and repeat the same reasonings used before. Linearity implies that  $\Lambda_D(\rho) = \sum_n p_n \Lambda_D(|\Psi_n\rangle\langle\Psi_n|)$ . Convexity implies that  $E[\Lambda_D(\rho)] \leq \sum_n p_n E[\Lambda_D(|\Psi_n\rangle\langle\Psi_n|)]$ , with the latter being the average entanglement of  $\Lambda_D(\rho)$ . The latter is in turn smaller or equal than  $\sum_n p_n (1-p)^2 E(|\Psi_n\rangle)$ , for (i) holds for any pure two-qubit state, so in particular also for all of the  $|\Psi_n\rangle$ . Therefore, we have

$$\begin{aligned} E[\Lambda_D(\rho)] &\leq \sum_n p_n (1-p)^2 E(|\Psi_n\rangle) = (1-p)^2 E\left(\sum_n p_n |\Psi_n\rangle\right) \\ &= (1-p)^2 E(\rho), \end{aligned} \quad (13)$$

where from the first to the second line of Eq. (13) we used the equivalence between the average entanglement and the entanglement itself for the optimal decomposition.

It is important to mention that, as was anticipated in Sec. I, bound (ii)—mathematically expressed in Eq. (13)—is directly connected to a universal law discovered in [41]. There, a universal bound on  $E[\Lambda(\rho)]$  was set for the particular case of  $E$  being the concurrence, but for  $\mathcal{E}$  any completely positive map. Bound (ii) thus generalizes [41] in terms of the allowed entanglement quantifier  $E$ . It restricts, however, the environment coupling to the particular case of depolarization dynamics and yields a slightly weaker bound when evaluated for  $E$  being the concurrence.

We also stress that bounds (i) and (ii) are optimal with respect to the class of states and entanglement quantifiers we deal with. This can be seen directly by showing an entanglement quantifier saturating the bound for at least one state, which was done already in Ref. [31], where it was shown that the most resistant negativities of evolved GHZ states tend to  $(1-p)^N$  times their initial value in the small  $p$  or large  $N$  limits.

### B. Dephasing

For individual dephasing environments the results are similar to the previously obtained. In this case we prove the following:

(iii) The entanglement of an initially generalized GHZ-diagonal state  $\rho$  subject to local dephasing  $\Lambda_{PD}$  is bounded from above as

$$E[\Lambda_{PD}(\rho)] \leq (1-p)^N E(\rho). \quad (14)$$

The strategy to show this is the same as in (i), but now with the dephasing channel (7). In this fashion, we first identify the terms in  $\Lambda_{PD}(\rho_0)$  for a single general GHZ state  $\rho_0$

$= |\psi_0^+(\alpha, \beta)\rangle\langle\psi_0^+(\alpha, \beta)|$  that sum up to some separable state. We notice then that only the first Kraus operator of  $\mathcal{E}_{PD}$  (proportional to the identity operator) yields a non(necessarily)-separable state upon application on  $|\psi_0^+(\alpha, \beta)\rangle$ . All other terms contain at least one diagonal projector,  $|0\rangle\langle 0|$  or  $|1\rangle\langle 1|$ , that eliminates the coherences of  $|\psi_0^+(\alpha, \beta)\rangle$  and takes it to some diagonal and hence separable matrix. Therefore, a decomposition of the final state is

$$\Lambda_{PD}(\rho_0) = (1-p)^N \rho_0 + [1 - (1-p)^N] \rho_{\text{sep}}, \quad (15)$$

where again  $\rho_{\text{sep}}$  is some fully separable density operator and where the procedure just used can be applied once more to any generalized GHZ state  $|\psi_k^+(\alpha, \beta)\rangle$ . The rest of the proof follows thus as in (i) from the linearity of the channel and the convexity property of the entanglement quantifier.

We should note that, in contrast to the depolarizing channel, the dephasing channel is indeed basis dependent—it does not commute with all local unitaries. Even though any pure two-qubit state is of generalized GHZ form in its Schmidt basis, this basis is not necessarily the one in which the dephasing channel is defined. Thus (iii) does not generalize to all two-qubit states as was the case with (i). We also stress that again bound (iii) is tight, as follows from a similar argument as for bounds (i) and (ii).

### C. Thermal bath

For the last example of environment models treated in this work we prove the following:

(iv) The entanglement of any initially generalized GHZ state  $\rho_k = |\psi_k^+(\alpha, \beta)\rangle\langle\psi_k^+(\alpha, \beta)|$  subject to local thermal baths  $\Lambda_T$  with an average excitation  $\bar{n}$  in the bath modes is bounded from above as

$$\begin{aligned} E[\Lambda_T(\rho_k)] &\leq \left[ |\alpha|^2 \left(1 - \frac{\bar{n}}{2\bar{n}+1} p\right)^{N-\kappa} \left(1 - \frac{\bar{n}+1}{2\bar{n}+1} p\right)^\kappa \right. \\ &\quad \left. + |\beta|^2 \left(1 - \frac{\bar{n}}{2\bar{n}+1} p\right)^\kappa \left(1 - \frac{\bar{n}+1}{2\bar{n}+1} p\right)^{N-\kappa} \right] E_{\text{max}}, \end{aligned}$$

where  $\kappa$  is the number of ones in the binary string  $k$ , i.e., the number of excitations in the state  $|k\rangle$ , and  $E_{\text{max}}$  is the maximal value that the entanglement quantifier  $E$  can take.

As before, we start by considering the final state in the operator sum representation of the channel acting on the initial state and identify terms that cancel to separable contributions to the state. Similarly to (iii) we find that only terms with the two Kraus operators  $K_0$  or  $K_2$ , out of all four of Eq. (9), applied to all of the qubits constitute the contributions that render a non(necessarily)-separable state. That is,  $\Lambda_T(\rho_k) = \sum_{j_1, \dots, j_N=0,2} K_{j_1} \otimes \dots \otimes K_{j_N} \rho_k K_{j_1}^\dagger \otimes \dots \otimes K_{j_N}^\dagger + \lambda_{\text{sep}} \rho_{\text{sep}}$ . For brevity we refer with  $\lambda_{\text{ent}} \rho_{\text{ent}}$  to the first, in general non-separable term. By virtue of the entanglement quantifier's

convexity the final entanglement is bounded from above, as in the previous cases, by the probability  $\lambda_{\text{ent}}$  of the state contribution  $\rho_{\text{ent}}$  in the first term in the convex sum, times its entanglement, and, trivially, by the maximal entanglement

$$E[\Lambda_T(\rho_k)] \leq \lambda_{\text{ent}} E(\rho_{\text{ent}}) \leq \lambda_{\text{ent}} E_{\text{max}}. \quad (16)$$

The probability  $\lambda_{\text{ent}}$  is given by the trace of the first term. It is

$$\begin{aligned} \lambda_{\text{ent}} &= \text{Tr} \left[ \sum_{j_1, \dots, j_N=0,2} K_{j_1}^\dagger K_{j_1} \otimes \dots \otimes K_{j_N}^\dagger K_{j_N} \rho_0 \right] \\ &= \text{Tr} \left[ \prod_{i=1}^N \left( \left( 1 - \frac{\bar{n}}{2\bar{n}+1} p \right) |0_i\rangle\langle 0_i| + \left( 1 - \frac{\bar{n}+1}{2\bar{n}+1} p \right) |1_i\rangle\langle 1_i| \right) \rho_k \right] \\ &= \langle \psi_k^\pm(\alpha, \beta) | \prod_{i=1}^N \left[ \left( 1 - \frac{\bar{n}}{2\bar{n}+1} p \right) |0_i\rangle\langle 0_i| + \left( 1 - \frac{\bar{n}+1}{2\bar{n}+1} p \right) |1_i\rangle\langle 1_i| \right] | \psi_k^\pm(\alpha, \beta) \rangle \\ &= \left[ |\alpha|^2 \left( 1 - \frac{\bar{n}}{2\bar{n}+1} p \right)^{N-\kappa} \left( 1 - \frac{\bar{n}+1}{2\bar{n}+1} p \right)^\kappa + |\beta|^2 \left( 1 - \frac{\bar{n}}{2\bar{n}+1} p \right)^\kappa \left( 1 - \frac{\bar{n}+1}{2\bar{n}+1} p \right)^{N-\kappa} \right], \end{aligned} \quad (17)$$

where the explicit definitions of  $K_0$  and  $K_2$  in Eq. (9) were used in the second equality.

The scaling factor in the bound of (iv) includes  $\kappa$  as well as the amplitudes  $\alpha$  and  $\beta$  and thus still incorporates details of the initial states. In order to arrive at a scaling factor that only includes details of the thermal environment and, of course, the number of qubits, we maximize the expression over all states  $|\psi_k^\pm(\alpha, \beta)\rangle$  which we are focusing on. The solution is reached, for instance, in the limit  $|\alpha|^2 \rightarrow 1$  and  $\kappa = 0$ . This allows us to get a larger, but state independent, scaling factor such that we can rephrase (iv):

(v) The entanglement of any initially generalized GHZ state  $\rho_k = |\psi_k^\pm(\alpha, \beta)\rangle\langle \psi_k^\pm(\alpha, \beta)|$  subject to local thermal baths  $\Lambda_T$  with an average excitation  $\bar{n}$  in the bath modes is bounded from above as

$$E[\Lambda_T(\rho_k)] \leq \left( 1 - \frac{\bar{n}}{2\bar{n}+1} p \right)^N E_{\text{max}}. \quad (18)$$

It is worth stressing that for this kind of noise model we are restricted to the (pure) generalized GHZ states. Also, notice that for different temperatures, and thus different mean numbers of bath excitations  $\bar{n}$ , bounds (iv) and (v) range from  $E[\Lambda_T(\rho_k)] \leq [|\alpha|^2(1-p)^\kappa + |\beta|^2(1-p)^{N-\kappa}] E_{\text{max}}$  for bound (iv) and a trivial bound for (v),  $E[\Lambda_T(\rho_k)] \leq E_{\text{max}}$ , in the purely dissipative zero-temperature limit with  $\bar{n}=0$  (amplitude damping channel), to  $E[\Lambda_T(\rho_k)] \leq (1 - \frac{p}{2})^N E_{\text{max}}$  for both bounds in the purely diffusive infinite-temperature limit  $\bar{n} \rightarrow \infty$ .

### V. BEYOND THE GENERALIZED GHZ STATES: COMPARISON TO RANDOM STATES

The simplicity of the obtained bounds raises the question of how dependent they are on the specific choice of initial states. Some of the above bounds are in this sense more

general than others, but all feature an exponential scaling in the number  $N$  of qubits, which renders their entanglement more fragile for increasingly many qubits.

In order to verify up to what extent the bounds are representative to a larger class of states than only to the ones they were actually derived for, we consider initial random pure states uniformly distributed over the entire Hilbert space [42]. As a simple convex entanglement quantifier we choose negativity evaluated for a bipartition of the  $N$  subsystems. For systems with varying number of constituents, we generate a sample of 10 000 initial states, evolve it under the three considered incoherent dynamics, and calculate the negativity as a function of the probability  $p$  of an incoherent event.

In general, we observe that such states violate the above bounds, more drastically as we increase the number of parties  $N$ . Figures 1 and 2 show examples of this violation. There, the average negativities  $\text{Neg}(p)$  [normalized to their initial value  $\text{Neg}(0)$ ] of a sample of 10 000 initial pure states undergoing individual depolarization is plotted for systems of different size  $N$  (blue crosses). For comparison, they also contain the bound  $(1-p)^N$  (red solid line) and the specific case of the balanced GHZ state  $|\psi_0^\pm(1/\sqrt{2}, 1/\sqrt{2})\rangle$  (dotted line). Necessarily, the latter always lies below the bound  $(1-p)^N$  and approaches it as  $N$  grows. The average normalized negativity, however, violates the  $(1-p)^N$  exponential-decay scaling law, in particular for larger  $N$ . The gray shadings along the vertical direction represent the histogram of the samples' normalized negativities in grayscale (bin population reflected by saturation, i.e., black the maximum and white the minimum) [43]. Figure 1 presents the data for negativity evaluated for the most balanced partition of  $N/2$  versus  $N/2$ , where the violation is apparent for six qubits. Contrary, Fig. 2 shows the least-balanced partition of one versus  $N-1$  qubits, where a violation appears for five qubits.

With the observation that initial random states violate the bounds, we can consider an extension of these bounds in such a fashion that these states are included as well without

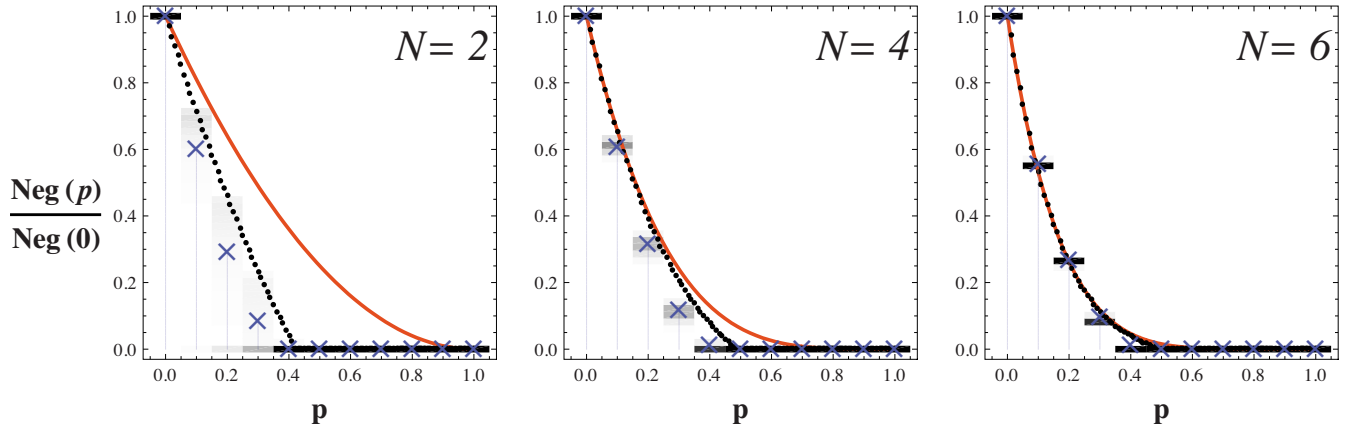


FIG. 1. (Color online) Normalized negativities of most balanced bipartitions as a function of  $p$  for systems of different size  $N$  undergoing individual depolarization. Red solid lines corresponds to the bound  $(1-p)^N$ , the dotted lines to the balanced GHZ state  $|\psi_0^+(1/\sqrt{2}, 1/\sqrt{2})\rangle$ , which always lie below the bound  $(1-p)^N$  and approach it as  $N$  grows, and the blue crosses to the average over 10 000 initially random pure states distributed uniformly over the system Hilbert space. Gray shadings are the distributions of the normalized negativities around their mean values (bin population reflected by saturation, i.e., dark is the maximum and white the minimum). For  $N=6$  the average normalized negativity no longer lies below the bound  $(1-p)^N$  (see text).

losing the scaling behavior, i.e., an entanglement upper bound that decreases with increasing  $N$  for all the states. Figure 3 suggests that this is *not* possible—the mean value of the normalized negativity, of the least-balanced bipartition, at a given evolution step (dephasing at  $p=0.3$ ) grows with increasing  $N$ . In this plot, the full line represents the bound  $(1-p)^N$ , while the dots stand for the numerical values obtained by a similar sampling as explained above. However, due to the big computational effort of sampling many states of large  $N$ , we rely on the negativity concentration effect for large dimensional systems under incoherent dynamics [43–45] in order to pick just few typical states. The sample sizes are displayed in Table I.

A persistently increasing mean negativity, which we confirmed numerically up to  $N=14$  qubits, can thus not be bounded from above by *any* nontrivially decreasing function. This also implies that the exponential fragility of GHZ-type states is not typical, as it is not a decisive feature present in the ensemble of randomly chosen initial states.

The increasing entanglement robustness for random states is observed for all channels here scrutinized. However, it

only takes place for highly unbalanced partitions, where the entanglement between few qubits with the remaining system is probed. For more balanced partitions, although the bounds presented in Sec. IV are violated (as shown in Fig. 1), we observe a stronger entanglement decay of random states for increasing number of subsystems. Thus, if an improved bound exists which applies also to random states, it must take into account the relative partition sizes in a many-body state.

## VI. CONCLUSIONS

In this paper we have considered generalized GHZ and GHZ-diagonal states evolving under the action of natural decoherence processes. For these states we have derived analytical upper bounds to the entanglement decay and shown that the fragility of their entanglement increases exponentially as the size of the system grows and independent of relative partition sizes. A comparison to random pure initial states under the same dynamics manifests the exponential

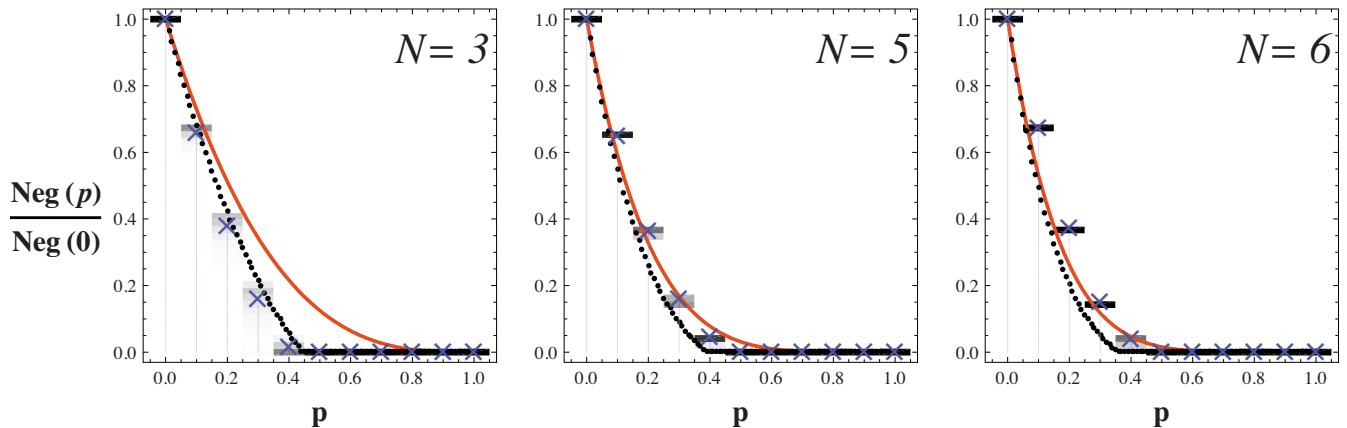


FIG. 2. (Color online) Same as Fig. 1, but for the least-balanced bipartitions. In this case the violation of the  $(1-p)^N$  scaling law already appears at  $N=5$  (see text).

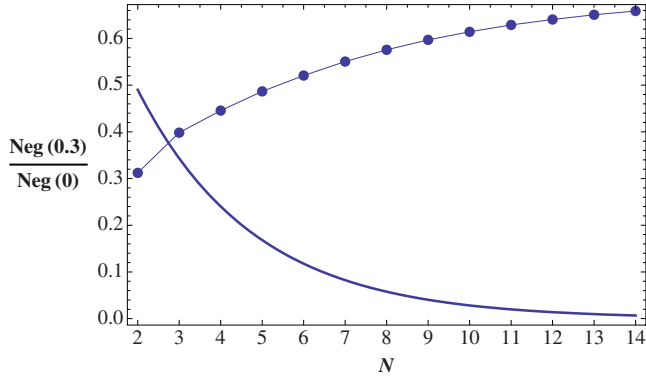


FIG. 3. (Color online) Mean normalized negativity for the most unbalanced partition as a function of the system size, for a fixed value of the probability  $p=0.3$ , and random initial states undergoing the action of the dephasing channel (circles, joined by straight lines as a guide to the eyes). Bound (iii) is plotted as a solid line.

fragility of entanglement toward decoherence to be a distinct feature of GHZ-type states.

We stress once more that the class of entanglement quantifiers considered in the calculations is very general. Thus, the present conclusions are not only valid for bipartite but also for any genuine multipartite entanglement the system can present. Moreover, most of the entanglement quantifiers aiming at the usefulness of a given quantum state for certain applications that involve local operations fall into the category of quantifiers adopted in the present work [46]. In this way, our results bound the GHZ states' usefulness for most of quantum communicational and computational tasks.

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#### APPENDIX: ENTANGLED CONTRIBUTION IN THE OPERATOR SUM REPRESENTATION OF DEPOLARIZING

In the proof of (i), the application of the depolarizing channel  $\Lambda_D$  to the initial state  $\rho_0 = |\psi_0^+(\alpha, \beta)\rangle\langle\psi_0^+(\alpha, \beta)|$

TABLE I. Sample size used in the numerical calculation of Fig. 3.

$N$	Sample size
2–7	10000
8–10	5000
11	1560
12	100
13	10
14	1

yields, in the operator sum representation using Eq. (6), two contributions. The application of operator  $\sigma_1$  or  $\sigma_2$  to any of the qubits gives a separable contribution, whereas a nonseparable contribution can exclusively appear from the application of only  $\sigma_0=1$  or  $\sigma_3$  to all the qubits

$$\sum_{j_1, \dots, j_N=0,3} s_{j_1} \dots s_{j_N} \sigma_{j_1} \otimes \dots \otimes \sigma_{j_N} \rho_0 \sigma_{j_1} \otimes \dots \otimes \sigma_{j_N}, \quad (\text{A1})$$

with respective prefactors  $s_0=1-3p/4$  and  $s_3=p/4$ . An even number of parity changes due to  $\sigma_3$  (and identity operators everywhere else) leaves the state effectively unchanged,  $|\psi_0^+(\alpha, \beta)\rangle\langle\psi_0^+(\alpha, \beta)|$ , whereas an odd number changes the parity,  $|\psi_0^-(\alpha, \beta)\rangle\langle\psi_0^-(\alpha, \beta)|$ . The corresponding prefactors are then accordingly given by the sum over all even and odd powers of  $s_3$  and  $s_0$ , respectively, weighted with their occurrence

$$\lambda_+ = \sum_{M=0, \text{even}}^N \binom{N}{M} \left(1 - \frac{3p}{4}\right)^{N-M} \left(\frac{p}{4}\right)^M, \quad (\text{A2})$$

$$\lambda_- = \sum_{M=1, \text{odd}}^N \binom{N}{M} \left(1 - \frac{3p}{4}\right)^{N-M} \left(\frac{p}{4}\right)^M. \quad (\text{A3})$$

Their difference is what determines the contribution to the remaining coherences, by virtue of the binomial theorem

$$\begin{aligned} \lambda_+ - \lambda_- &= \sum_{M=0}^N \binom{N}{M} \left(1 - \frac{3p}{4}\right)^{N-M} \left(-\frac{p}{4}\right)^M \\ &= (1-p)^N. \end{aligned} \quad (\text{A4})$$

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