Parametric down-conversion from a wave-equation approach: Geometry and absolute brightness

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(Received 22 July 2008; revised manuscript received 6 January 2009; published 30 April 2009)

Using the approach of coupled wave equations, we consider spontaneous parametric down conversion (SPDC) in the narrow-band regime and its relationship to classical nonlinear processes such as sum-frequency generation. We find simple expressions in terms of mode-overlap integrals for the absolute pair-production rate into single-spatial modes and simple relationships between the efficiencies of the classical and quantum processes. The results, obtained with Green function techniques, are not specific to any geometry or nonlinear crystal. The theory is applied to both degenerate and nondegenerate SPDC. We also find a time-domain expression for the correlation function between filtered signal and idler fields.

DOI: 10.1103/PhysRevA.79.043835 PACS number(s): 42.50.Ct, 42.50.Dv

I. INTRODUCTION

Spontaneous parametric down-conversion (SPDC) has become a workhorse technique for generation of photon pairs and related states in quantum optics. Improvements in both nonlinear materials [1] and down-conversion geometries have led to a steady growth in the brightness of these sources [2–9].

Applications of the bright sources include fundamental tests of quantum mechanics, quantum communications, quantum information processing, and quantum metrology [10–13]. Although down-conversion sources typically have bandwidths of order 10^{11} Hz, for the brightest sources even the output in a few-MHz window can be useful for experiments. This permits a new application, the interaction of down-conversion pairs with atoms, ions, or molecules. Indeed, sources for this purpose have been demonstrated [14]. Many modern applications use single-spatial-mode collection either for improved spatial coherence, to take advantage of fiber-based technologies, or to separate the source and target for experimental convenience.

Remarkably, despite the importance of bright single-spatial-mode sources, general methods for calculating the absolute brightness of such a source are not found in the literature. By absolute brightness, we mean the number of pairs per second that are collected for specified beam shapes, pump power, filters, and crystal characteristics. A number of calculations study the dependence of brightness on parameters such as beam widths or collection angles, but these typically give only relative brightness: the final results contain an unknown multiplicative constant [15,16]. While useful for optimizing a given source, they are less helpful when designing new sources. A recent paper computes the absolute brightness for a specific geometry: Gaussian beams in the thin-crystal limit [17].

In this paper, we calculate the absolute brightness for narrow-band paraxial sources. The results are quite general; for example, they apply equally well to crystals with spatial or temporal walk-off, for non-Gaussian beams, etc. The Green function approach we use is well suited to describing the temporal features of the down-conversion pairs and we are able to predict the time correlations in a particularly simple way.

Perhaps of greatest practical importance, we derive very simple relationships between the efficiency of classical parametric processes and their corresponding quantum parametric processes. For example, in any given geometry the efficiencies of sum-frequency generation and spontaneous parametric down-conversion are proportional. This allows the use of existing classical calculations and/or experiments with classical nonlinear optics to predict the brightness of quantum sources.

The paper is organized as follows. In Sec. II, we describe briefly the variety of theoretical treatments that have been applied to parametric down-conversion and our reasons for making a new calculation. In Sec. III we describe the formalism we use, based on an abstract paraxial wave equation and Green function solutions. In Sec. IV we calculate the absolute brightness and efficiencies for nondegenerate and degenerate parametric down-conversion and the corresponding classical processes. In Sec. VI we summarize the results.

II. BACKGROUND

The characteristics of parametric down-conversion light have been calculated in a number of different ways. Kleinman [18] used a Hamiltonian of the form

$$H' = -\frac{1}{3}\int d^3x \mathbf{E}\cdot\chi{:}\mathbf{E} \tag{1}$$

and the Fermi "golden rule" to derive emission rates as a function of frequency and angle. Zeldovich and Klyshko [19] proposed to use a mode expansion and calculate pair rates treating the quantum process as a classical parametric amplifier seeded by vacuum noise. Detailed treatment along these lines is given in [20,21]. The problem of collection into defined spatial modes was not considered; indeed the works emphasize that the *total* rate of emission is *independent* of pump focusing.

After the observation of SPDC temporal correlations by Burnham and Weinberg [22], Mollow [23] described detectable field correlation functions (coincidence distributions) in terms of source-current correlations and Green functions of the wave equation. This Heisenberg-picture calculation derived absolute brightness for multimode collection, e.g., for detectors of defined area at defined positions. It did not give brightness for single-mode collection or a connection to classical nonlinear processes. Hong and Mandel [24] used a mode expansion to compute correlation functions based on the Heisenberg-picture evolution and an interaction Hamiltonian of the form

$$H_I = \frac{1}{2}\int d^3x \chi^{(2)}_{ijk} E_i E_j E_k. \tag{2}$$

As with Mollow's calculation, they find singles and pair-detection rates, but only for multimode detection [25]. Ghosh, *et al.* [26] used the same Hamiltonian in a Schrödinger-picture description, truncating the time evolution at first order to derive a "two-photon wave function." This last method has become the most popular description of SPDC, including work on efficient collection into single-spatial modes [15,16]. Many works along these lines are cited in reference [15]. Recently, Ling, *et al.* [17] calculated the absolute emission rate based on a similar interaction Hamiltonian and a Gaussian-beam mode expansion. In this way, they are able to calculate the absolute pair rate for non-degenerate SPDC in a uniform thin crystal into Gaussian collection modes. As described in Sec. V C, our calculation agrees with that of Ling *et al.* [17] while also treating other crystal geometries, general beam shapes, and degenerate SPDC.

Notable differences among the calculations include Heisenberg vs Schrödinger picture and calculating in direct space vs inverse space via a mode expansion. While they are of course equivalent, Heisenberg-picture calculations are easier to compare to classical optics, while Schrödinger-picture calculations are more similar to the state representations in quantum information. As our goal is in part to connect classical and quantum efficiencies, we use the Heisenberg picture. Also, we note that the Schrödinger-picture "two-photon wave function" has a particular pathology: the first-order treatment of time evolution means the Schrödinger-picture state is not normalized and never contains more than two down-conversion photons. While this is not a problem for calculation of relative brightness or pair distributions [27,28], it does prevent calculation of absolute brightness. The choice of inverse vs real-space calculation is also one of convenience: for large angles in birefringent media or detection in momentum space, plane waves are the "natural" basis for the calculation. However, most bright sources use paraxial geometries and collection into defined spatial modes, e.g., the Gaussian modes of optical fibers. In these situations, the advantages of a mode expansion disappear, while the local nature of the $\chi^{(2)}$ interaction makes real space more "natural." Thus we opt for a real-space calculation.

Our treatment of SPDC is based on coupled wave equations: a standard approach for multiwave mixing in nonlinear optics [29]. The calculations are done in the Heisenberg picture so that the evolution of the quantum fields is exactly parallel to that of the classical fields described by nonlinear optics. This allows the reuse of well-known classical calculations such as those by Boyd and Kleinman [30]. As in the approach of Mollow, we use Green functions to describe the propagation and find results that are not specific to any particular crystal or beam geometry. Unlike Mollow's calculation, we work with a paraxial wave equation (PWE). This allows us to simply relate the classical and quantum processes through momentum reversal, which takes the form of complex conjugation in the PWE.

We focus on narrow-band parametric down-conversion, for which the results are particularly simple. By narrow band, we mean that the bandwidths of the pump and of the collected light are much less than the bandwidth of the SPDC process, as set by the phase-matching conditions. This includes recent experiments with very narrow filters [14], but also a common configuration in SPDC, in which the down-conversion bandwidth is $\sim$10 nm while the filter bandwidths are $<$1 nm.

III. FORMALISM

A. Description of propagation

We are interested in the envelopes $\mathcal{E}_\pm$ for forward- and backward-directed parts of the quantum field $E^{(+)}(t,\mathbf{x})=(\mathcal{E}_+\exp[+ikz]+\mathcal{E}_-\exp[-ikz])\exp[-i\omega t]$, where k is the average wave number and ω is the carrier frequency. These propagate according to a paraxial wave equation

$$\mathcal{D}_\pm \mathcal{E}_\pm = \mathcal{S}_\pm, \tag{3}$$

where $\mathcal{D}_\pm$ is a differential operator and $\mathcal{S}_\pm$ is a source term (later due to a $\chi^{(2)}$ nonlinearity).

The formal (retarded) solution to Eq. (3) is

$$\mathcal{E}_\pm(x) = \mathcal{E}_{0\pm}(x) + \int d^4x' \mathcal{G}_\pm(x;x')\mathcal{S}_\pm(x'), \tag{4}$$

where x is the four-vector $(t,\mathbf{x})$, $\mathcal{E}_{0\pm}(x)$ is a solution to the source-free ($\mathcal{S}=0$) equation, and $\mathcal{G}_\pm$ are the time-forward Green functions defined by

$$\mathcal{D}_\pm \mathcal{G}_\pm(x;x') = \delta^4(x-x'),$$

$$\mathcal{G}_\pm(x;x') = 0 \quad t<t'. \tag{5}$$

For illustration, we consider the PWE for which

$$\mathcal{D}_\pm \equiv \nabla_T^2 \pm 2ik(\partial_z \pm v_g^{-1}\partial_t), \tag{6}$$

$$\mathcal{S}_\pm = \frac{\omega^2}{c^2\varepsilon_0}\mathcal{P}_\pm^{(\mathrm{NL})}. \tag{7}$$

Here ∇_T^2 is the transverse Laplacian, $k=n(\omega)\omega/c$ is the wave number, $v_g \equiv \partial\omega/\partial k_z$ is the group velocity, and $\mathcal{P}^{(\mathrm{NL})}$ is the envelope for the nonlinear polarization.

We note that $\mathcal{D}_\pm$ is invariant under translations of x and that time reversal $t\to -t$ is equivalent to direction reversal and complex conjugation, i.e., $\mathcal{D}_\pm \to \mathcal{D}_\mp^*$. The results we obtain will be valid for any equation obeying these symmetries. In particular, the results will also apply to propagation with dispersion and/or spatial walk-off, which can be included by adding other time and/or spatial derivatives to $\mathcal{D}$.

From the symmetries of $\mathcal{D}_\pm$, it follows that the Green functions depend only on the difference $x-x'$ and that $\mathcal{G}_+(t,\mathbf{x};t',\mathbf{x}')=\mathcal{G}_-^*(t,\mathbf{x}',t',\mathbf{x})$. Also, the time-backward (or "advanced") Green functions $\mathcal{H}_\pm$, defined by

$$\mathcal{D}_\pm\mathcal{H}_\pm(x;x')=\delta^4(x-x'),$$

$$\mathcal{H}_\pm(x;x')=0 \quad t>t', \tag{8}$$

obey $\mathcal{H}_\pm(x;x')=\mathcal{G}_\pm^*(x',x)$.

B. Boundary and initial value problems

If the value of the field is known on a plane $z=z_{\rm src}$, the field downstream of that plane is

$$\mathcal{E}_\pm(x)=\beta_z\int d^4x'\mathcal{G}_\pm(x;x')\mathcal{E}_\pm(x')\delta(z'-z_{\rm src}), \tag{9}$$

where $\beta_z\equiv\pm 2ik$. Similarly, if the field is known at an initial time $t=t_0$, the field later is

$$\mathcal{E}_\pm(x)=\beta_t\int d^4x'\mathcal{G}_\pm(x;x')\mathcal{E}_\pm(x')\delta(t'-t_0), \tag{10}$$

where $\beta_t=2ik/v_g$. Similar relationships hold for the advanced Green functions. If the field is known in some plane $z=z_0$ downstream, then

$$\mathcal{E}_\pm(x)=\beta_z^*\int d^4x'\mathcal{H}_\pm(x;x')\mathcal{E}_\pm(x')\delta(z'-z_0)$$

$$=\beta_z^*\int d^4x'\mathcal{E}_\pm(x')\delta(z'-z_0)\mathcal{G}_\pm^*(x';x), \tag{11}$$

while if the field is known at some time t_f in the future,

$$\mathcal{E}_\pm(x)=\beta_t^*\int d^4x'\mathcal{H}_\pm(x;x')\mathcal{E}_\pm(x')\delta(t'-t_f)$$

$$=\beta_t^*\int d^4x'\mathcal{E}_\pm(x')\delta(t'-t_f)\mathcal{G}_\pm^*(x';x). \tag{12}$$

C. Quantization

The field envelopes are operators which obey the equal-time commutation relation

$$[\mathcal{E}(\mathbf{x},t),\mathcal{E}^\dagger(\mathbf{x}',t)]=A_\gamma^2\delta^3(\mathbf{x}'-\mathbf{x}), \tag{13}$$

where $A_\gamma\equiv\sqrt{\hbar\omega/2nn_g\varepsilon_0}$ is a photon units scaling factor and $n_g\equiv c/v_g$ is the group index. For narrow-band fields, $A_\gamma^{-2}\langle\mathcal{E}^\dagger\mathcal{E}\rangle$ describes a photon number density and $v_gA_\gamma^{-2}\langle\mathcal{E}^\dagger\mathcal{E}\rangle$ and $v_{gs}v_{gi}A_{\gamma i}^{-2}A_{\gamma s}^{-2}\langle\mathcal{E}_s^\dagger\mathcal{E}_i^\dagger\mathcal{E}_i\mathcal{E}_s\rangle$ describe single and pair fluxes. We find the unequal-time commutation relation from Eq. (10),

$$[\mathcal{E}(x),\mathcal{E}^\dagger(x')]_{t>t'}=\beta_tA_\gamma^2\mathcal{G}(x;x'), \tag{14}$$

so that $\langle 0|\mathcal{E}(x)\mathcal{E}^\dagger(x')|0\rangle=\beta_tA_\gamma^2\mathcal{G}(x;x')$ for $t>t'$. For the PWE, $A_\gamma^{-2}v_g=2nc\varepsilon_0/\hbar\omega$ and $\beta_tA_\gamma^2=i\hbar\omega^2/c^2\varepsilon_0$.

To calculate singles rates, we will need to evaluate expressions of the form $\langle\mathcal{E}\mathcal{E}^\dagger\rangle$. For this, a useful expression is derived in Appendix A: Eq. (A2),

$$\langle\mathcal{E}(x)\mathcal{E}^\dagger(x')\rangle=\frac{2\hbar n\omega^3}{c^3\varepsilon_0}\int d^4x''\delta(z''-z_0)\mathcal{G}^*(x'';x)\mathcal{G}(x'';x'). \tag{15}$$

Here z_0 is any plane down stream of x and x'.

D. Single-spatial modes

A single-spatial mode $M_\pm(\mathbf{x})$ is a time-independent solution to the source-free wave equation $\mathcal{D}_\pm M_\pm(\mathbf{x})=0$. $M_\pm^*(\mathbf{x})$ is the corresponding momentum-reversed solution $\mathcal{D}_\mp M_\pm^*(\mathbf{x})=0$. We assume the normalization $\int d^3x|M_\pm(\mathbf{x})|^2\delta(z)=1$. For single-mode collection, it will be convenient to define the projection of a field $\mathcal{E}(x)$ onto the mode M as

$$\mathcal{E}_M(t)\equiv\int d^3xM^*(\mathbf{x})\delta(z-z_0)\mathcal{E}(x) \tag{16}$$

(here and below, the $+/-$ propagation direction is the same for $\mathcal{E},M$). Here z_0 is some plane of interest and $\mathcal{E}_M(t)$ describes the magnitude of the field component in this plane. Similarly, if the envelope is constant, the field distribution is

$$\mathcal{E}(x)=\mathcal{E}_M(t)M(\mathbf{x}). \tag{17}$$

The optical power is (MKS units) $P_M(t)=2nc\varepsilon_0\int d^3x|\mathcal{E}(t,x)|^2\delta(z-z_0)=2nc\varepsilon_0|\mathcal{E}_M(t)|^2$.

Given an upstream source $\mathcal{S}(x)$, the M component of the generated field is

$$\mathcal{E}_M(t)=\int d^3xd^4x'M^*(\mathbf{x})\delta(z-z_0)\mathcal{G}(x;x')\mathcal{S}(x'). \tag{18}$$

If the source is time independent, then Eq. (11) and the time-translation symmetry of $\mathcal{G}$ imply

$$\mathcal{E}_M(t)=\frac{1}{\beta_z}\int d^3x'M^*(\mathbf{x}')\mathcal{S}(x'). \tag{19}$$

Similarly, if a product $\mathcal{E}_1(x_1)\mathcal{E}_2(x_2)$ is given by a constant pair source $\mathcal{S}^{(2)}(x)$ as

$$\mathcal{E}_1(x_1)\mathcal{E}_2(x_2)=\int d^4x'\mathcal{G}_1(x_1;x')\mathcal{G}_2(x_2;x')\mathcal{S}^{(2)}(x'), \tag{20}$$

then the time-integrated mode-projected component is

$$\int dt_1\mathcal{E}_{1M_1}(t_1)\mathcal{E}_{2M_2}(t_2)=\frac{1}{\beta_{1z}\beta_{2z}}\int d^3x'M_1^*(\mathbf{x}')M_2^*(\mathbf{x}')\mathcal{S}^{(2)}(x'). \tag{21}$$

E. Coupled wave equations

We now introduce a $\chi^{(2)}$ nonlinearity, which produces a nonlinear polarization that appears as a source term in the propagation equations. We consider three fields, "signal," "idler," and "pump," with carrier frequencies $\omega_s, \omega_i, \omega_p$ and wave numbers k_s, k_i, k_p, respectively. The respective field envelopes $\mathcal{E}_s, \mathcal{E}_i, \mathcal{E}_p$ evolve according to

$$\mathcal{D}_p \mathcal{E}_p = \omega_p^2 g \mathcal{E}_s \mathcal{E}_i \exp[i\Delta kz],$$

$$\mathcal{D}_s \mathcal{E}_s = \omega_s^2 g \mathcal{E}_p \mathcal{E}_i^\dagger \exp[-i\Delta kz],$$

$$\mathcal{D}_i \mathcal{E}_i = \omega_i^2 g \mathcal{E}_p \mathcal{E}_s^\dagger \exp[-i\Delta kz], \tag{22}$$

where $g = -4m(\mathbf{x})d/c^2$, d is the effective nonlinearity, equal to half the relevant projection of $\chi^{(2)}$, and $\Delta k \equiv k_p - k_s - k_i$ is the wave-number mismatch. The dimensionless function $m(\mathbf{x})$ describes the distribution of $\chi^{(2)}$. For example, in a periodically poled material it alternates between ± 1. We can take $\Delta k = 0$ without loss of generality, as the phase oscillation can be incorporated directly in the envelopes. The propagation directions $(\pm)$ will be omitted unless needed for clarity. Note that for transparent materials $\chi^{(2)}$ is real and $\chi^{(2)}(\omega_p;\omega_s+\omega_i) = \chi^{(2)}(\omega_s;\omega_p-\omega_i) = \chi^{(2)}(\omega_i;\omega_p-\omega_s)$.

First-order perturbation theory is sufficient to describe situations in which pairs are produced. For example, if $\mathcal{E}_{0s}, \mathcal{E}_{0i}, \mathcal{E}_{0p}$ are source-free solutions, then

$$\mathcal{E}_s = \mathcal{E}_{0s} + \omega_s^2 \int d^4x' \mathcal{G}_s(x;x') g(x') \mathcal{E}_{0p}(x') \mathcal{E}_{0i}^\dagger(x') + O(g^2) \tag{23}$$

and similar expressions for $\mathcal{E}_i, \mathcal{E}_p$ are sufficient to give the lowest-order contribution to the pair-detection rate $W^{(2)} \propto \langle \mathcal{E}_s^\dagger \mathcal{E}_i^\dagger \mathcal{E}_i \mathcal{E}_s \rangle$. Higher-order expansions would be necessary for double-pair production, etc.

F. Narrow-band frequency filters

In most down-conversion experiments, some sort of frequency filter is used. Assuming this filter is linear and stationary, the field reaching the detector is

$$\mathcal{E}^{(F)}(t) = \int dt' F(t-t') \mathcal{E}(t') + G(t-t') \mathcal{E}_{\text{res}}(t'). \tag{24}$$

Here $\mathcal{E}_{\text{res}}$ is a reservoir field required to maintain the field commutation relations. Assuming the reservoir is in the vacuum state, it will not produce detections and can be ignored. Defining $H_F(t_i,t_s) \equiv \langle \mathcal{E}_i^{(F_i)}(t_i) \mathcal{E}_s^{(F_s)}(t_s) \rangle$, the fields that leave the filter obey

$$H_F(t_i,t_s) = \int dt' dt'' F_i(t_i - t') F_s(t_s - t'') \langle \mathcal{E}_i(t') \mathcal{E}_s(t'') \rangle. \tag{25}$$

In the narrowband case, i.e., when the correlation time between signal and idler is much less than the time scale of the impulse response functions, we can take $\langle \mathcal{E}_i(t') \mathcal{E}_s(t'') \rangle \approx \mathcal{A} \delta(t' - t'')$, where the constant $\mathcal{A} \equiv \int dt_i \langle \mathcal{E}_i(t_i) \mathcal{E}_s(t_s) \rangle$. We find

$$H_F(t_i,t_s) \approx \mathcal{A} \int dt' F_i(t_i - t') F_s(t_s - t') \equiv \mathcal{A} f(t_s - t_i). \tag{26}$$

With this, we see that the flux of pairs is

$$W^{(2)}(t_s - t_i) = \frac{4 n_s n_i c^2 \varepsilon_0^2}{\hbar^2 \omega_s \omega_i} |\mathcal{A} f(t_s - t_i)|^2, \tag{27}$$

with a total coincidence rate of

$$W^{(2)} = \int dt_i W^{(2)}(t_s - t_i) = \frac{n_s n_i c^2 \varepsilon_0^2}{\hbar^2 \omega_s \omega_i} |\mathcal{A}|^2 \int dt_i |2 f(t_s - t_i)|^2$$
$$\equiv \frac{n_s n_i c^2 \varepsilon_0^2}{\hbar^2 \omega_s \omega_i} |\mathcal{A}|^2 \Gamma_{\text{eff}}. \tag{28}$$

We note that

$$\Gamma_{\text{eff}} = \frac{2}{\pi} \int d\Omega T_s(\Omega) T_i(-\Omega), \tag{29}$$

where $T_{s,i}(\Omega) \equiv |\int dt \exp[i\Omega t] F_{s,i}(t)|^2$ are the signal and idler filter transmission spectra, respectively. For this reason we refer to Γ_{eff} as the effective linewidth (in angular frequency) for the combined filters. Also important will be the singles rate

$$W^{(1)} = A_{\gamma s}^{-2} v_{gs} \langle [\mathcal{E}_s^{(F_S)}(t_s)]^\dagger \mathcal{E}_s^{(F_S)}(t_s) \rangle$$
$$= \frac{2 n_s c \varepsilon_0}{\hbar \omega_s} \int dt' dt'' F_s^*(t_s - t') F_s(t_s - t'') \langle \mathcal{E}_s^\dagger(t') \mathcal{E}_s(t'') \rangle$$
$$\approx \frac{2 n_s c \varepsilon_0}{\hbar \omega_s} \mathcal{C} \int dt' |F_s(t_s - t')|^2$$
$$\equiv \frac{n_s c \varepsilon_0}{2\hbar \omega_s} \mathcal{C} \Gamma_{\text{eff},s}, \tag{30}$$

where $\mathcal{C} \equiv \int dt' \langle \mathcal{E}_s^\dagger(t') \mathcal{E}_s(t'') \rangle$. $\Gamma_{\text{eff},s}$ is the effective linewidth for the signal filter.

IV. RESULTS

With the calculational tools described above, we now demonstrate the central results of this paper. We first express the efficiency of continuous-wave sum-frequency generation (SFG) in terms of a mode-overlap integral. This effectively reduces the nonlinear optical problem to three uncoupled propagation problems. We then show that the efficiency of parametric down-conversion in the same medium is proportional to the SFG efficiency for modes with the same shapes but opposite propagation direction. The constant of proportionality is found, allowing calculations of absolute efficiency based either on material properties such as $\chi^{(2)}$ or measured second-harmonic generation (SHG) efficiencies. Similarly, the singles production efficiency is related to difference-frequency generation (DFG) and the collection efficiency is calculated. The same quantities for the degenerate case are also found.

A. Sum-frequency generation

We consider first the process of SFG for undepleted signal and idler and no input pump. Signal and idler are constant and come from single-modes

$$\mathcal{E}_{M_p}(t_p) = -\mathcal{E}_{M_i}\mathcal{E}_{M_s}\frac{4\omega_p^2 d}{c^2\beta_{z,p}}\int d^3x' M_p^*(x')m(x')M_i(x')M_s(x')$$
$$\equiv -\mathcal{E}_{M_i}\mathcal{E}_{M_s}\frac{4\omega_p^2 d}{c^2\beta_{z,p}} I_{\mathrm{SFG}}. \tag{31}$$

The conversion efficiency is

$$Q_{\mathrm{SFG}} \equiv \frac{P_{M_p}}{P_{M_s}P_{M_i}} = \frac{8\omega_p^4 n_p d^2|I_{\mathrm{SFG}}|^2}{n_s n_i c^5 \varepsilon_0|\beta_{z,p}|^2} = \frac{2\omega_p^2 d^2}{c^3\varepsilon_0 n_p n_s n_i}|I_{\mathrm{SFG}}|^2. \tag{32}$$

The efficiency of a cw single-mode source is thus proportional to the spatial overlap of the pump, signal, and idler modes, weighted by the nonlinear coupling g.

B. Nondegenerate parametric down-conversion

Next we consider the process of parametric down-conversion. Using Eq. (23), we can calculate to first order in g the correlation function

$$\langle\mathcal{E}_i(x_i)\mathcal{E}_s(x_s)\rangle = \omega_s^2\int d^4x'\mathcal{G}_s(x_s,x')$$
$$\times\langle\mathcal{E}_{0,i}(x_i)\mathcal{E}_{0,i}^{\dagger}(x')\rangle g(x')\mathcal{E}_{0,p}(x')$$
$$= i\frac{\hbar\omega_i^2\omega_s^2}{c^2\varepsilon_0}\int d^4x'\mathcal{G}_s(x_s,x')\mathcal{G}_i(x_i,x')g(x')\mathcal{E}_{0,p}(x'). \tag{33}$$

For constant pump and single-mode collection we have

$$\mathcal{A}_{M_iM_s} \equiv \int dt_s\langle\mathcal{E}_{M_i}(t_i)\mathcal{E}_{M_s}(t_s)\rangle$$
$$= \frac{i\hbar\omega_s\omega_i d}{c^2\varepsilon_0 n_s n_i}\mathcal{E}_{M_p}\int d^3x' M_s^*(x')M_i^*(x')m(x')M_p(x')$$
$$\equiv \frac{i\hbar\omega_s\omega_i d}{c^2\varepsilon_0 n_s n_i}\mathcal{E}_{M_p} I_{DC}. \tag{34}$$

We note that $I_{DC}=I_{\mathrm{SFG}}^*$. Also, the conjugate modes describe backward-propagating fields, as if the source fields were sent through the nonlinear medium in the opposite direction. Thus if we want to know the brightness of down-conversion when all beams are propagating to the left, it is sufficient to calculate (or measure) the efficiency of up-conversion when all beams are propagating to the right. Using Eqs. (32) and (34) we find

$$|\mathcal{A}_{M_iM_s}|^2 = \frac{\hbar^2\omega_i^2\omega_s^2}{4c^2\varepsilon_0^2 n_s n_i\omega_p^2}P_p Q_{\mathrm{SFG}}. \tag{35}$$

C. Brightness

We can now consider the brightness of the filtered single-mode source. The rate of detection of pairs is

$$W^{(2)} = \frac{n_s n_i c^2\varepsilon_0^2}{\hbar^2\omega_s\omega_i}|\mathcal{A}|^2\Gamma_{\mathrm{eff}} = \Gamma_{\mathrm{eff}}\frac{\omega_i\omega_s}{4\omega_p^2}P_p Q_{\mathrm{SFG}}. \tag{36}$$

This simple expression is the first main result. The rate of pairs is simply the joint collection bandwidth Γ_{eff}, times the ratio of frequencies, times the pump power, times the up-conversion efficiency Q_{SFG}. Note that the last quantity can be calculated if the mode shapes and $\chi^{(2)}(\mathbf{x})$ are known, for example, in the paper of Boyd and Kleinman [30] or simulated for more complicated situations. Most importantly, it is directly measurable.

D. Difference-frequency generation

We now consider the classical situation in which pump and signal beam are injected into the crystal and idler is generated. We will see that this directly measurable process is related to the singles generation rate by parametric down-conversion. The generated idler is

$$\mathcal{E}_i(x) = \omega_i^2\int d^4x'\mathcal{G}_i(x;x')g(x')\mathcal{E}_{0p}(x')\mathcal{E}_{0s}^*(x').$$

If pump and signal are from modes M_P, M_S, respectively, we find

$$\mathcal{E}_i(x) = -\frac{4\omega_i^2 d}{c^2}\mathcal{E}_{M_p}(t_p)\mathcal{E}_{M_s}^*(t_s)\int d^4x'\mathcal{G}_i(x;x')m(x')$$
$$\times M_p(x')M_s^*(x'). \tag{37}$$

The total power generated is $P_i=2cn_i\varepsilon_0\int d^3x_i\delta(z_i-z_0)|\mathcal{E}_i(x_i)|^2$, where z_0 indicates a plane downstream of the generation. We find

$$P_i = \frac{2\omega_i^2 d^2}{c^3\varepsilon_0 n_s n_i n_p}P_p P_s\int d^3x_i\delta(z_i - z_0)$$
$$\times\left|\beta_{z,i}\int d^4x'\mathcal{G}_i(x_i;x')m(x')M_p(x')M_s^*(x')\right|^2$$
$$\equiv P_p P_s\frac{2\omega_i^2 d^2}{c^3\varepsilon_0 n_s n_i n_p}|I_{\mathrm{DFG}}^{(s)}|^2$$
$$\equiv P_p P_s Q_{\mathrm{DFG}}. \tag{38}$$

E. Singles rates in parametric down-conversion

We can find the rate of detection of singles in the mode M_S by Eq. (30) and using Eq. (A2),

$$\mathcal{C} = \int dt_s \langle \mathcal{E}^\dagger_{M_S}(x_s)\mathcal{E}_{M_S}(x'_s)\rangle$$
$$= \int dt_s d^3x_s d^3x'_s M_s(\mathbf{x}_s)M_s^*(\mathbf{x}'_s)\delta(z_s - z_0)\delta(z'_s - z_0)$$
$$\times\langle \mathcal{E}_s^\dagger(x_s)\mathcal{E}_s(x'_s)\rangle$$
$$= \frac{|\mathcal{E}_{Mp}|^2\omega_s^4}{|\beta_{z,s}|^2}\int d^3x d^3x' M_s(\mathbf{x})g(\mathbf{x})M_p^*(\mathbf{x})$$
$$\times\langle \mathcal{E}_{0i}(x)\mathcal{E}_{0i}^\dagger(x')\rangle M_s^*(\mathbf{x}')g(\mathbf{x}')M_p(\mathbf{x}')$$
$$= \frac{2\hbar\omega_i\omega_s^2 d^2}{c^3 n_s^2 n_i \varepsilon_0}|\mathcal{E}_{Mp}|^2\int d^4x'' \delta(z''-z_0)$$
$$\times\left|\beta_{i,z}\int d^3x G_i(x'';x)M_s^*(\mathbf{x})m(\mathbf{x})M_p(\mathbf{x})\right|^2, \quad (39)$$

so that

$$W^{(1)} = \frac{c\omega_i\omega_s}{32 n_p n_s n_i \varepsilon_0}P_p|I_{\mathrm{DFG}}^{(s)}|^2\Gamma_{\mathrm{eff},s} = \frac{\omega_s}{4\omega_i}\Gamma_{\mathrm{eff},s}P_p Q_{\mathrm{DFG}}^{(s)}. \quad (40)$$

F. Conditional efficiency

The conditional efficiency for the idler (probability of collecting the idler, given that the signal was collected) is

$$\eta_s \equiv \frac{W^{(2)}}{W_s^{(1)}} = \frac{\Gamma_{\mathrm{eff}}}{\Gamma_s}\frac{|I_{\mathrm{SFG}}|^2}{|I_{\mathrm{DFG}}^{(s)}|^2} \quad (41)$$

G. Degenerate processes

Up to this point, we have discussed only nondegenerate processes, i.e., those in which the signal and idler fields are distinct and do not interfere. This is always the case for type-II down-conversion and will be the case for type-I down-conversion if the frequencies and/or directions of propagation are significantly different. We now consider degenerate processes, in which there is only one down-converted field (signal).

The above discussion is modified only slightly. The signal and pump evolve by

$$\mathcal{D}_p\mathcal{E}_p = \frac{1}{2}\omega_p^2 g\mathcal{E}_s\mathcal{E}_s,$$

$$\mathcal{D}_s\mathcal{E}_s = \omega_s^2 g\mathcal{E}_p\mathcal{E}_s^\dagger. \quad (42)$$

H. Second harmonic generation

The calculation of SHG proceeds exactly as in sum-frequency generation, except for the factor of 1/2 and with all idler variables replaced by signal variables. Thus we find

$$P_p = P_s^2 Q_{\mathrm{SHG}}, \quad (43)$$

where

$$Q_{\mathrm{SHG}} = \frac{\omega_p^2 d^2}{2c^3\varepsilon_0 n_p n_s^2}|I_{\mathrm{SHG}}|^2 \quad (44)$$

and

$$I_{\mathrm{SHG}} \equiv \int d^3x M_p^*(\mathbf{x})m(\mathbf{x})M_s(\mathbf{x})M_s(\mathbf{x}). \quad (45)$$

I. Average parametric gain

The other classical process of interest is parametric amplification of the signal by the pump. The first-order solution for the signal field is

$$\mathcal{E}_s = \mathcal{E}_{0s} + \omega_s^2\int d^4x' \mathcal{G}_s(x;x')g(x')\mathcal{E}_{0p}(x')\mathcal{E}_{0s}^\dagger(x') \equiv \mathcal{E}_{0s} + \mathcal{E}_{1s}. \quad (46)$$

The signal power at the output is

$$P_s = 2n_s c\varepsilon_0\int d^3x_s \delta(z_s - z_0)|\mathcal{E}_s(x_s)|^2$$
$$= 2n_s c\varepsilon_0\int d^3x_s \delta(z_s - z_0)(|\mathcal{E}_{0,s}(x_s)|^2$$
$$+ 2\mathrm{Re}[\mathcal{E}_{0,s}(x_s)\mathcal{E}_{1,s}^*(x_s)] + |\mathcal{E}_{1,s}(x_s)|^2). \quad (47)$$

The first term is the input signal power $P_{0,s}$, the second term depends on the relative phase $\phi_p - 2\phi_s$, and the last term is the phase-independent contribution to the gain, an experimentally accessible quantity. We have

$$\overline{\delta P} \equiv \langle P_s - P_{0,s}\rangle_{\phi_s}$$
$$= 2n_s c\varepsilon_0\int\int d^3x_s\delta(z_s - z_0)|\mathcal{E}_{1,s}(x_s)|^2$$
$$= \frac{8\omega_s^2\varepsilon_0 d^2}{cn_s}|\mathcal{E}_{0s}|^2|\mathcal{E}_p|^2\int d^4x_s\delta(z_s - z_0)$$
$$\times\left|\beta_{z,s}\int d^3x' \mathcal{G}_s(x_s;x')m(\mathbf{x}')M_p(\mathbf{x}')M_s^*(\mathbf{x}')\right|^2$$
$$\equiv P_{0s}P_p\frac{2\omega_s^2 d^2}{c^3 n_s^2 n_p\varepsilon_0}|I_{\mathrm{APG}}|^2$$
$$\equiv P_{0s}P_p Q_{\mathrm{APG}}. \quad (48)$$

J. Degenerate parametric down-conversion

Next we consider the process of degenerate parametric down-conversion, for which

$$\mathcal{E}_s = \mathcal{E}_{0s} + \omega_s^2\int d^4x' \mathcal{G}_s(x;x')g(x')\mathcal{E}_{0p}(x')\mathcal{E}_{0s}^\dagger(x'). \quad (49)$$

We find the correlation function

$$\langle\mathcal{E}_s(x_s)\mathcal{E}_s(x'_s)\rangle = \omega_s^2\int d^4x''\mathcal{G}_s(x_s,x'')$$
$$\times\langle\mathcal{E}_{0,s}(x'_s)\mathcal{E}_{0,s}^\dagger(x'')\rangle g(x'')\mathcal{E}_{0,p}(x''), \quad (50)$$

at which point it is clear that the only difference from the nondegenerate case of Eq. (33) will be the replacement of idler variables with signal variables. We find

$$W^{(2)} = \Gamma_{\text{eff}}\frac{\omega_s^2}{4\omega_p^2}P_p Q_{\text{SHG}} = \frac{\Gamma_{\text{eff}}}{16}P_p Q_{\text{SHG}}. \tag{51}$$

K. Singles rates (degenerate)

As before, we can find the rate of detection of singles in the mode M_S by Eq. (30) and

$$\begin{aligned}\mathcal{C} &= \int dt_s' \langle \mathcal{E}^\dagger_{M_S}(t_s')\mathcal{E}_{M_S}(t_s'')\rangle \\ &= \int dt_s' d^3x_s' d^3x_s' M_s(\mathbf{x}_s')M_s^*(\mathbf{x}_s'')\delta(z_s'-z_0)\delta(z_s''-z_0) \\ &\quad\times\langle \mathcal{E}_s^\dagger(x_s')\mathcal{E}_s(x_s'')\rangle \\ &= \frac{|\mathcal{E}_{Mp}|^2\omega_s^4}{|\beta_{z,s}|^2}\int d^3x' d^3x'' M_s(\mathbf{x}')g(\mathbf{x}')M_p^*(\mathbf{x}') \\ &\quad\times\langle \mathcal{E}_{0i}(x')\mathcal{E}_{0i}^\dagger(x'')\rangle M_s^*(\mathbf{x}'')g(\mathbf{x}'')M_p(\mathbf{x}'') \\ &= \frac{2\hbar\omega_s^3 d^2}{c^3 n_s^3\varepsilon_0}|\mathcal{E}_{Mp}|^2\int d^4x''\,\delta(z''-z_0) \\ &\quad\times\left|\beta_{s,z}\int d^3x G_s(x'';x)M_s^*(\mathbf{x})m(\mathbf{x})M_p(\mathbf{x})\right|^2.\end{aligned} \tag{52}$$

The singles rate is thus

$$W_s^{(1)} = \frac{1}{4}\Gamma_{\text{eff},s}P_p Q_{\text{APG}}^{(s)}. \tag{53}$$

L. Conditional efficiency (degenerate)

The conditional efficiency is

$$\eta_s \equiv \frac{W^{(2)}}{W_s^{(1)}} = \frac{\Gamma_{\text{eff}}}{\Gamma_{\text{eff},s}}\frac{|I_{\text{SHG}}|^2}{|I_{\text{APG}}^{(s)}|^2} \tag{54}$$

V. EXAMPLE CALCULATIONS

We now illustrate the preceding general results with a few special cases. We first calculate the overlap integral for copropagating Gaussian beams. This allows us to (1) compare our results to the classical results of Boyd and Kleinman [30], (2) predict absolute brightness for an important geometry, type-II collinear down-conversion in quasi-phase-matched material. Also, we compare to a recent calculation of absolute brightness for a specific geometry by Ling *et al.* [17].

We consider collinear frequency-degenerate type-II parametric down-conversion (PDC) with circular Gaussian beams for signal, idler, and pump. We take mode shape functions

$$M_m(\mathbf{x}) = \sqrt{\frac{k_m z_R}{\pi}}\frac{1}{q}e^{ik_m z}e^{ik_m r^2/2q}, \tag{55}$$

where $m \in \{s,i,p\}$, r is the radial component of $\mathbf{x}$, and $q \equiv z - iz_R$, where z_R is the Rayleigh range, assumed equal for all beams. We assume a periodically poled material in which $\chi^{(2)}(z)$ alternates with period $2\pi/Q$ so we approximate $m(\mathbf{x}) \approx \exp[iQz]d_{\text{eff}}/d$. From Eq. (31) we find

$$\begin{aligned}I_{\text{SFG}} &= \sqrt{\frac{k_p k_s k_i z_R^3}{\pi^3}}\int_{-L/2}^{L/2} dz' \frac{e^{-i\Delta k z'}}{q|q|^2} \\ &\quad\times\int 2\pi r' dr' e^{-i(k_p/q^* - k_s + k_i/q)r'^2/2} \\ &= \frac{2i}{k_+}\sqrt{\frac{k_p k_s k_i z_R^3}{\pi}}\int_{-L/2}^{L/2} dz' \frac{e^{-i\Delta k z'}}{(z'-iz_R)(R_k z' + iz_R)},\end{aligned} \tag{56}$$

where $\Delta k \equiv k_p - k_s - k_i - Q$, $R_k \equiv k_-/k_+$, and $k_\pm \equiv k_p \pm (k_s + k_i)$. In terms of the dimensionless variables $\kappa \equiv \Delta k L$, $\zeta \equiv z'/L$, and $\zeta_R \equiv z_R/L$ we find

$$I_{\text{SFG}} = \frac{2i}{k_+}\sqrt{\pi k_p k_s k_i z_R}\Upsilon, \tag{57}$$

where

$$\Upsilon \equiv \frac{\zeta_R}{2\pi}\int_{-1/2}^{1/2} d\zeta \frac{e^{-i\kappa\zeta}}{(\zeta - i\zeta_R)(R_k\zeta + i\zeta_R)}. \tag{58}$$

From Eq. (32) the up-conversion efficiency is then

$$Q_{\text{SFG}} = \frac{8\pi\omega_p^2}{c^3\varepsilon_0 n_p n_s n_i}\frac{k_p k_s k_i}{k_+^2}z_R d_{\text{eff}}^2|\Upsilon|^2. \tag{59}$$

A. Boyd and Kleinman [30]

With this expression we can compare our results to those of Boyd and Kleinman [30] for the case of second-harmonic generation. As that calculation does not include quasiphase matching, we take $Q=0$ and then for any reasonable phase-matching we have $n_p \approx n_s$, $k_p \approx 2k_s$ and thus $k_- = R_k = 0$, $k_+ \approx 2k_p$. We note that for $R_k = 0$, Υ becomes equal to the function H of Boyd and Kleinman for zero absorption and walk-off angle. We find

$$P_p = \frac{4\pi k_s\omega_s^2}{c^3\varepsilon_0 n_s^2 n_p}z_R d^2|\Upsilon|^2 P_s^2. \tag{60}$$

Boyd and Kleinman find in Eqs. (2.16), (2.17), and (2.20)–(2.24)

$$P_2 = \frac{128\pi^2\omega_1^2}{c^3 n_1^2 n_2}d^2 P_1^2 L k_1 \frac{2\pi^2 z_R}{L}|H|^2 \tag{61}$$

or

$$P_2 = \frac{256\pi^4 k_1\omega_1^2}{c^3 n_1^2 n_2}z_R d^2|H|^2 P_1^2. \tag{62}$$

When converting this expression to MKS units, $d^2 \to d^2/64\pi^3\varepsilon_0$ and we see that the two calculations agree.

B. Type-II collinear brightness

Next we make a numerical calculation for frequency-degenerate type-II SPDC, a geometry of current interest for generation of entangled pairs, for example. The integral Υ must be evaluated numerically. For a 1 cm crystal of periodically-poled potassium titanyl phosphate (PPKTP) and a vacuum wavelength $\lambda_s=\lambda_i=800$ nm we have $(n_s,n_i,n_p)=(1.844,1.757,1.964)$ and $d_{\rm eff}=2.4$ pm/V so that $R_k=0.04$ and the maximum of $(z_R/L)|\Upsilon|^2\approx 0.054$ occurs at $\kappa\approx -3.0$, $\zeta_R\approx 0.18$. We find $Q_{\rm SFG}=2.0\times 10^{-3}$ W^{-1}. Used as a photon-pair source, this same crystal and geometry would yield by Eq. (36),

$$W^{(2)} = \Gamma_{\rm eff} P_p \frac{Q_{\rm SFG}}{16} \tag{63}$$

or a pair generation efficiency of $Q_{\rm SFG}/16=0.8$ pairs (s mW MHz)$^{-1}$. Note that $\Gamma_{\rm eff}$ is the filter bandwidth in angular frequency.

C. Ling *et al.* [17]

Recently, Ling *et al.* [17] calculated the absolute emission rate into Gaussian modes in the thin-crystal limit of (nonperiodically poled) nonlinear material. They arrive to a down-conversion spectral brightness of

$$\frac{dR(\omega_s)}{d\omega_s} = \left(\frac{d_{\rm eff}\alpha_s\alpha_i E_p^0 \Phi(\Delta k)}{c}\right)^2 \frac{\omega_s\omega_i}{2\pi n_s n_i}, \tag{64}$$

where R is the pair collection rate and

$$\Phi(\Delta k) \equiv \int dz \int dy dx e^{i\Delta \mathbf{k}\cdot\mathbf{r}} U_p(\mathbf{r}) U_s(\mathbf{r}) U_i(\mathbf{r}). \tag{65}$$

Here U_m describe the mode shapes of the form $U_m(\mathbf{r}) = e^{ik_m z} e^{-(x^2+y^2)/W_m^2}$ and $\alpha_m=\sqrt{2/\pi W_m^2}$ are normalization constants. The field E_p^0 is defined such that $|E_p^0|^2=2\alpha_p^2 P_p/\varepsilon_0 n_p c$, where P_p is the pump power giving

$$\frac{dR(\omega_s)}{d\omega_s} = \frac{\omega_s\omega_i d^2}{\pi c^3 \varepsilon_0 n_p n_s n_i} P_p |\alpha_p\alpha_s\alpha_i \Phi(\Delta k)|^2. \tag{66}$$

Assuming the output is collected with narrow-band filters of transmission $T_s(\omega_s)$, $T_i(\omega_i)$ for signal and idler, respectively, the integrated rate is

$$R = \frac{dR(\omega_s)}{d\omega_s} \int d\Omega T_s(\omega_p/2+\Omega) T_i(\omega_p/2-\Omega), \tag{67}$$

where we have assumed $dR(\omega_s)/d\omega_s$ constant over the width of the filters. For comparison, using Eqs. (32) and (36), we find

$$W^{(2)} = \frac{\omega_s\omega_i d^2}{2c^3\varepsilon_0 n_p n_s n_i} P_p |I_{\rm SFG}|^2 \Gamma_{\rm eff}. \tag{68}$$

Then with Eq. (29) and noting that in the thin-crystal limit $|I_{\rm SFG}|=|\alpha_p\alpha_s\alpha_i\Phi(\Delta k)|$, we see that the two results are identical.

VI. CONCLUSIONS

Using the approach of coupled wave equations, familiar from nonlinear optics, we have calculated the absolute brightness and temporal correlations of spontaneous parametric down-conversion in the narrow-band regime. The results are obtained with a Green function method and are generally valid within the paraxial regime. We find that efficiencies of SFG and SPDC can be expressed in terms of mode-overlap integrals and are proportional for corresponding geometries. Also, we find pair time correlations in terms of signal and idler filter impulse response functions. Results for both degenerate and nondegenerate SPDC are found. Comparisons to classical calculations by Boyd and Kleinman [30] and to a recent calculation by Ling *et al.* [17] show the connection to classical nonlinear optics and golden rule–style brightness calculations, while considerably generalizing the latter. We expect these results to be important both for designing SPDC sources, as the results of well-known classical calculations can be used, and for building and optimizing such sources.

ACKNOWLEDGMENTS

We thank A. Cerè, F. Wolfgramm, G. Molina, A. Haase, and N. Piro for helpful discussions. This work was supported by the Spanish MEC under the ILUMA project (Ref. FIS2008-01051), the Consolider-Ingenio 2010 Project "QOIT," and by Marie Curie RTN "EMALI."

APPENDIX A: ALTERNATE PROPAGATOR

We can use Eqs. (12) and (13) to express the propagator as

$$\begin{aligned}\langle \mathcal{E}(x)\mathcal{E}^\dagger(x')\rangle &= |\beta_t|^2 \int d^4x'' d^4x''' \delta(t''-t_f)\delta(t'''-t_f) \\ &\quad \times \langle \mathcal{E}(x'')\mathcal{E}^\dagger(x''')\rangle \mathcal{G}^*(x'';x)\mathcal{G}(x''';x') \\ &= |A_\gamma \beta_t|^2 \int d^4x'' \delta(t''-t_f) \mathcal{G}^*(x'';x)\mathcal{G}(x'';x').\end{aligned} \tag{A1}$$

Noting that $\int d^4x''\delta(t''-t_f)\mathcal{G}^*(x'';x)\mathcal{G}(x'';x') = v_g \int d^4x''\delta(z''-z_0)\mathcal{G}^*(x'';x)\mathcal{G}(x'';x')$, we find

$$\langle \mathcal{E}(x)\mathcal{E}^\dagger(x')\rangle = \frac{2\hbar n \omega^3}{c^3\varepsilon_0} \int d^4x'' \delta(z''-z_0) \mathcal{G}^*(x'';x)\mathcal{G}(x'';x'). \tag{A2}$$

APPENDIX B: LORENTZIAN FILTER

A common filter has a Lorentzian transfer function and an exponential impulse response

$$F(\tau) = \frac{\Gamma}{2}\theta(\tau)\exp[-\Gamma\tau/2]. \tag{B1}$$

The spectral transmission is $T(\Omega)=\Gamma^2/(\Gamma^2+4\Omega^2)$, i.e., unit transmission for constant $\mathcal{E}$, a full width at half maximum of $\Delta\Omega_{\rm FWHM}=\Gamma$, and an area $\int d\Omega T(\Omega)=\pi\Gamma/2$. If we put a filter of this sort in each arm, the output has

$$f(t_s - t_i) = \frac{\Gamma_s\Gamma_i}{4}\int dt'\,\theta(t_s - t')\,\theta(t_i - t')\exp[-\Gamma_i(t_i - t')/2] \times\exp[-\Gamma_s(t_s - t')/2] \quad \text{(B2)}$$

or

$$f(\tau) = \frac{\Gamma_s\Gamma_i}{2(\Gamma_s + \Gamma_i)}\begin{cases}\exp[-\Gamma_s\tau/2] & \tau > 0\\ \exp[\Gamma_i\tau/2] & \tau < 0.\end{cases} \quad \text{(B3)}$$

The effective bandwidth is

$$\Gamma_{\mathrm{eff}} = 4\int d\tau |f(\tau)|^2 = \frac{\Gamma_s\Gamma_i}{\Gamma_s + \Gamma_i}. \quad \text{(B4)}$$

It is worth noting that in the limit $\Gamma_s\to\infty$ (the limit of a broadband filter in the signal beam or, in practical terms, not having a filter there at all), filter becomes

$$f(t_s - t_i) = \frac{\Gamma_i}{2}\begin{cases}0 & t_i < t_s\\ \exp[-\Gamma_i(t_i - t_s)/2] & t_i > t_s.\end{cases} \quad \text{(B5)}$$

That is, the idler photon will always arrive later and with a distribution (after the signal arrival) that is precisely the transfer function of the idler-beam filter. Another interesting limit is for matched filters, $\Gamma_s=\Gamma_i=\Gamma$. Then we find

$$f(t_s - t_i) = \frac{\Gamma}{4}\exp[-\Gamma|t_s - t_i|/2]. \quad \text{(B6)}$$

Note that for $\Gamma_s\to\infty$, the detection rate is $|\mathcal{A}|^2\Gamma_i/4$, i.e., proportional to the idler filter bandwidth Γ_i. The reverse $s\leftrightarrow i$ is also true, of course. From this we can get an idea of the conditional efficiency. The rate for filtered signal with any idler is proportional to

$$\Gamma_s \geq \frac{\Gamma_i\Gamma_s}{\Gamma_s + \Gamma_i}. \quad \text{(B7)}$$

For example, putting matched filters $\Gamma_s=\Gamma_i=\Gamma$ will give a rate proportional to $\Gamma_i\Gamma_s/(\Gamma_s+\Gamma_i)$, i.e., half of the rate without the idler filter. This indicates that, of the signal photons that pass the signal filter, half of their "twin" idler photons do not pass the idler filter.

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