Group-theoretical approach to entanglement

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We present a universal description of quantum entanglement using group theory and noncommutative characteristic functions. It leads to reformulations of the separability problem, which allows us to generalize the latter, thus connecting the theory of entanglement and harmonic analysis. As an example, we translate and analyze the positivity of partial transpose criterion and a simple criterion for pure states into the group-theoretical language. We also show that when applied to finite groups, our formalism embeds the separability problem in a given dimension into a higher dimensional but highly symmetric one. Finally, our formalism reveals a connection between the very existence of entanglement and group noncommutativity.

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I. INTRODUCTION

Despite numerous attempts [1–7] and substantial time passed after its formulation [8], the problem of efficient description of entangled states of multipartite systems—the so-called, separability problem—remains still open. We recall [8–10] that a state ϱ of an N-partite system is called separable if it can be represented as a convex combination of product states:

$$\varrho = \sum_i p_i |\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| \otimes \cdots \otimes |\psi_i^{(N)}\rangle\langle\psi_i^{(N)}|. \tag{1}$$

Otherwise ϱ is called entangled. The importance of the separability problem lies in both practical applications, connected to the quantum information processing [10], as well as in the conceptual issues of quantum mechanics (see, e.g., Refs. [11,12]).

In this work we present a group-theoretical approach to the separability problem in finite dimensions. Our approach is based on a generalization of standard characteristic functions and results from the following two observations: (i) it is possible to identify a bipartite Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$ with a tensor product of representation spaces of two independent irreducible unitary representations of some suitable compact group G; (ii) one can then perform a noncommutative Fourier transform [13] and assign to each density matrix a unique function on $G \times G$, satisfying certain positivity conditions. This function is an analog of a classical characteristic function [14], but is defined on a generically non-Abelian group. We will call such functions "noncommutative characteristic functions." The group G will be called the "kinematical group" of an individual system.

Let us emphasize that for a generic quantum system the choice of the kinematical group is at this stage arbitrary—the *only* requirement is that G should possess irreducible unitary representations, matching the dimensionality of the system's Hilbert space. This freedom makes our approach very flexible and allows us to use as G, e.g., finite groups as well as Lie groups.

The potential significance of noncommutative characteristic functions for quantum mechanics has been first, up to our knowledge, pointed out in the physical literature by Gu in Ref. [15]. However, no investigation of the separability problem has been carried out there, as the work of Gu predates the seminal paper of Werner [8]. On the other hand, standard characteristic functions played a crucial role in solving the separability problem for Gaussian states [16], and in studying quantumness of states of harmonic oscillator (see Ref. [17] and references therein).

In the present work we use noncommutative characteristic functions to restate the separability problem in a new language. Although we do not present any new entanglement tests, our results offer a different point of view on this long-standing problem, and link it to harmonic analysis and group theory. In particular, we pose a generalized separability problem for noncommutative characteristic functions, which is an interesting mathematical problem in itself. As an example of the necessary generalized separability criterion, we reformulate the PPT criterion [1] in group-theoretical terms and show that it is connected to a certain simple operation on noncommutative characteristic functions. This connection is universal, and holds irrespectively of the group used. Apart from that, we translate one of the necessary and sufficient criteria for pure states. Using the freedom in the choice of the kinematical group, we examine an interesting case of finite kinematical groups, like permutation groups. This leads to an embedding of the separability problem in a given dimension into a higher dimensional one, with some specific symmetries, however. Quite interestingly, we also show a *purely formal* similarity of our formalism to local hidden variables (LHV) models [18]. Finally, the conceptually attracting feature of our approach is that it allows connecting the very existence of entanglement with the group's noncommutativity.

The work is organized as follows: in Sec. II we define the noncommutative characteristic functions, and review their properties (using Ref. [13] as the main mathematical reference). In Sec. III we use the developed formalism to reformulate the separability problem. Section IV is dedicated to the study of the PPT criterion. We show its "robustness" with respect to the change of the kinematical group. We also briefly examine one of the criteria for pure states there. In Sec. V we examine our separability criterion on finite groups. In Sec. VI we remark on the connection to the LHV models. Finally, in Sec. VII we sketch a possible reformulation of the mathematical language of quantum statistics, exposing the connection between entanglement and noncommutativity of the kinematical group.

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II. NONCOMMUTATIVE CHARACTERISTIC FUNCTIONS

We begin with presenting the general set up of our work. We consider an arbitrary compact group G (it may or may not be a Lie group) and let τ be any of its irreducible, unitary representation (in the sequel by representation we will always mean a unitary representation) acting in a Hilbert space $\mathcal{H}_\tau$. We study linear operators A acting in $\mathcal{H}_\tau$, and, in particular, density matrices ϱ. In the present work we fix G to be compact, as we study only finite dimensional systems here, and for a compact group all of its irreducible unitary representations are necessarily finite dimensional (see Refs. [15,19] for formalism of noncommutative characteristic functions on noncompact groups). Following Gu [15] (see also Ref. [20]), we assign to each operator A a continuous complex function ϕ_A on G through

$$\phi_A(g) \coloneqq \mathrm{tr}[A\tau(g)]. \tag{2}$$

For the particular case of a density matrix ϱ, the function ϕ_ϱ is a noncommutative analog of the usual Fourier transform of a probability measure—if we think of a state ϱ as of a quantum analog of a classical probability measure [21], then ϕ_ϱ is an analog of its characteristic function. Indeed, from the positivity of ϱ and Eq. (2), it follows that [15],

$$\int\int_{G\times G} dgdh\overline{f(g)}\phi_\varrho(g^{-1}h)f(h) \geqslant 0 \text{ for any } f \in L^1(G), \tag{3}$$

where dg is a normalized Haar measure on G, and the bar denotes complex conjugation [22]. Functions $\phi: G \to \mathbb{C}$ satisfying the above property are called positive definite on G [13]. Moreover, ϕ_ϱ is normalized

$$\phi_\varrho(e) = 1 \tag{4}$$

(e denotes the neutral element of G), which follows from the normalization of ϱ. Thus, ϕ_ϱ possesses all the features of a classical characteristic function, but it is defined on a non-Abelian group. Hence the term noncommutative characteristic function. It is the main object of our study.

Note that characteristic functions (2) are generally easy to calculate explicitly. For example, when $G=\mathrm{SU}(2)$ and $\tau=\tau_j$ carries spin j, they are polynomials of degree $2j$ in the group parameters. As an example, we calculate in the Appendix the characteristic function for the $3\otimes 3$ Horodecki's state from Ref. [3].

The crucial point for our approach is that, since τ is irreducible, one can invert the noncommutative Fourier transform (2) and recover operator A from its characteristic function [13,15]

$$A = \int_G dgd_\tau\phi_A(g)\tau(g)^\dagger, \quad d_\tau \coloneqq \dim \mathcal{H}_\tau. \tag{5}$$

The proof of (5) is most easily obtained by taking the matrix elements of both sides in some orthonormal basis of $\mathcal{H}_\tau$, and then by using the orthogonality of the matrix elements of τ, guaranteed by the Peter-Weyl theorem [23]. An interesting implication of Eq. (5) is that multiplication of operators corresponds to taking convolutions of the corresponding functions (2),

$$\phi_{AB} = d_\tau\phi_A * \phi_B, \tag{6}$$

where we define $f*f'(g) \coloneqq \int_G dhf(h)f'(gh^{-1}) = \int_G dhf(h^{-1}g)f'(h)$ (in the last step we substituted $h \to h^{-1}g$ and used the fact that $dg^{-1}=dg$ for compact groups [13]). In particular, state ϱ is pure if and only if

$$\phi_\varrho = d_\tau\phi_\varrho * \phi_\varrho. \tag{7}$$

Let us now focus on the space of all normalized, positive definite functions on G, i.e., the space of all continuous functions ϕ, satisfying the conditions (3) and (4). We denote this space by $\mathcal{P}_1(G)$. It is a convex subset of the space of all continuous functions on G, and the set of its extreme points we denote by $\mathcal{E}_1(G)$. The structure of $\mathcal{E}_1(G)$ is described by the Gelfand-Naimark-Segal construction (see, e.g., [13] or the note [24]): $\phi \in \mathcal{E}_1(G)$ if and only if there exists an irreducible unitary representation τ_ϕ of G and a normalized vector $\psi_\phi \in \mathcal{H}_\phi$ (the space of τ_ϕ) such that $\phi(g) = \langle\psi_\phi|\tau_\phi(g)\psi_\phi\rangle$. Thus, every $\phi \in \mathcal{E}_1(G)$ is a characteristic function of some pure state $\psi_\phi \in \mathcal{H}_\phi$. In particular, because of Eq. (7), it satisfies: $\phi = d_{\tau_\phi}\phi * \phi$.

Obviously, $\mathcal{P}_1(G)$ contains more functions than just characteristic functions of the type (2). To identify which $\phi \in \mathcal{P}_1(G)$ are characteristic functions of states, first note that from Eq. (2) it follows that

$$\phi_\varrho(g) = \sum_i p_i\langle\psi_i|\tau(g)\psi_i\rangle, \tag{8}$$

where we used any convex decomposition (for example, an eigenensemble) of ϱ: $\varrho = \Sigma_i p_i|\psi_i\rangle\langle\psi_i|$. From Eq. (8), we see that the decomposition of ϕ_ϱ into extreme points from $\mathcal{E}_1(G)$ contains only one, fixed representation τ. Conversely, let $\phi = \Sigma_i p_i\phi_i$ where $\mathcal{E}_1(G) \ni \phi_i = \langle\psi_i|\tau\psi_i\rangle$ for each i (such sums are finite, since all irreducible representations are finite-dimensional), then $\phi = \phi_\varrho$, where $\varrho \coloneqq \Sigma_i p_i|\psi_i\rangle\langle\psi_i|$.

Let us describe the inverse noncommutative Fourier transform (5) of the whole $\mathcal{P}_1(G)$. Note that since G is compact, the set of its irreducible representations is discrete, and we can label them by some natural index k. Then, any $\phi \in \mathcal{P}_1(G)$ defines, through the integral (5), a positive semidefinite operator $\varrho_k(\phi)$ for every irreducible representation τ_k. To prove it, note that for any $\psi \in \mathcal{H}_k$ (the space of τ_k) holds

$$\begin{aligned}\langle\psi|\varrho_k(\phi)\psi\rangle &= \int dgd_k\phi(g)\langle\psi|\tau_k(g)^\dagger\psi\rangle \\ &= \int\int dhdgd_k\phi(h^{-1}g)\langle\tau_k(h)^\dagger\psi|\tau_k(g)^\dagger\psi\rangle \\ &= \sum_{\mu=1}^{d_k}\int\int dhdgd_k\overline{\langle e_\mu|\tau_k(h)^\dagger\psi\rangle}\phi(h^{-1}g) \\ &\quad\times\langle e_\mu|\tau_k(g)^\dagger\psi\rangle \geqslant 0.\end{aligned} \tag{9}$$

In the second step above we changed the variables $g \to h^{-1}g$, used the invariance of dg and inserted $1=\int_G dh$, since the integrand did not depend on h. Then we inserted a unit matrix, decomposed with respect to an arbitrary basis $\{e_\mu\}$ of $\mathcal{H}_k$. However, generically $\varrho_k(\phi)$ is subnormalized as for a generic ϕ there appear all irreducible representations of G in the convex decomposition of ϕ into $\mathcal{E}_1(G)$. Hence, each $\phi \in \mathcal{P}_1(G)$ defines a positive semidefinite operator in the space $\oplus_k \mathcal{H}_k$, where the sum is over all irreducible representations of G, through

$$\varrho(\phi) := \bigoplus_k \varrho_k(\phi), \tag{10}$$

while each $\varrho_k(\phi)$ is given by Eq. (5). Only such a defined operator $\varrho(\phi)$ is normalized, which follows from the identity $\phi(g) = \Sigma_k \operatorname{tr}[\varrho_k(\phi)\tau_k(g)]$ [Ref. [13]]. Of course, if τ_k is not present in the decomposition of ϕ then, from the Peter-Weyl theorem, $\varrho_k(\phi) = 0$ [Ref. [23]].

Summarizing, states on an irreducible representation space $\mathcal{H}_\tau$ are in the one-to-one correspondence with functions on G satisfying the conditions (3), (4), and (8) [24]. The last condition ensures that in the decomposition (10) there appears only representation τ, and hence the operator given by Eq. (10) acts in the desired space and is normalized. The correspondence $\varrho \leftrightarrow \phi_\varrho$ may be heuristically viewed as a change of basis: $|e_\mu\rangle\langle e_\nu| \leftrightarrow \langle e_\nu|\tau(\cdot)e_\mu\rangle$.

The presented formalism is closely related to that of generalized coherent states [25]. Within the latter, every density matrix on $\mathcal{H}_\tau$ can be represented as: $\varrho = \int_{G/H} dx\, P_\varrho(x)|x\rangle\langle x|$, where H is an isotropy subgroup in representation τ of some fixed vector $\psi_0 \in \mathcal{H}_\tau$ and $|x\rangle$ are the corresponding coherent states. However, unlike noncommutative characteristic function, P-representation P_ϱ is generally nonunique and does not encode the positivity of a density matrix in a simple manner. For applications of generalized coherent states to the study of entanglement see, e.g., Refs. [20,26,27].

III. APPLICATION TO THE STUDY OF ENTANGLEMENT

Having established the formalism, we proceed to reformulate the separability problem in terms of noncommutative characteristic functions. Let us consider a bipartite, finite dimensional system, described by a Hilbert space $\mathcal{H} := \mathbb{C}^m \otimes \mathbb{C}^n$. At this point, we arbitrarily identify the spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ with two independent representation spaces $\mathcal{H}_\pi$, $\mathcal{H}_\tau$ of irreducible representations π, τ of some compact kinematical group G,

$$\mathbb{C}^m \equiv \mathcal{H}_\pi, \quad \mathbb{C}^n \equiv \mathcal{H}_\tau. \tag{11}$$

Of course, the group and the representations should be chosen to match the desired dimensions m, n. As we mentioned in the Introduction, this is the only constraint we impose on G.

The identification (11), although mathematically always possible and nonunique, may seem arbitrary from the physical point of view. For instance, for a given system we could have chosen another kinematical group G', possessing suitable representations. This freedom may in fact turn out to be a big advantage of the formalism, as the choice of G can be optimized in each practical case. There is also a "universal" kinematical group $G = \mathrm{SU}(2)$—since it possesses irreducible representations in all possible finite dimensions, it can serve as a kinematical group for all finite dimensional systems. The results of Sec. II imply then that we can describe through the formulas (2) and (5) *all* states in *all* finite dimensions in terms of noncommutative characteristic functions on SU(2). Thus, without a loss of generality, we may always treat our system as a system of (possibly artificial) independent spins $j_1 := (m-1)/2$ and $j_2 := (n-1)/2$.

Having done the identification (11), we can view the Hilbert space of the full system $\mathcal{H} = \mathcal{H}_\pi \otimes \mathcal{H}_\tau$ as the representation space of the product group $G \times G$ under the unitary representation $T := \pi \otimes \tau$, defined as

$$T(g_1, g_2) := \pi(g_1) \otimes \tau(g_2). \tag{12}$$

Representation T is irreducible as a representation of $G \times G$ [28] and moreover, every irreducible representation of $G \times G$ is of that form, up to a unitary equivalence [13]. Hence, we may view $G \times G$ as the kinematical group of the composite system. Since $G \times G$ is obviously compact, we can apply to it all the methods of Sec. II.

Let us consider a separable state ϱ on $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$, for which there exists a convex decomposition of the type (1): $\varrho = \Sigma_i p_i |u_i\rangle\langle u_i| \otimes |v_i\rangle\langle v_i|$. Then, from Eq. (2) we obtain that

$$\phi_\varrho(g_1, g_2) = \sum_i p_i \kappa_i(g_1)\eta_i(g_2), \tag{13}$$

where $\kappa_i(g_1) := \langle u_i|\pi(g_1)u_i\rangle$, $\eta_i(g_2) := \langle v_i|\tau(g_2)v_i\rangle$ are noncommutative characteristic functions from $\mathcal{P}_1(G)$, or more precisely from $\mathcal{E}_1(G)$. Conversely, a function of the form (13) defines a separable state through the integral (5), because

$$\int_{G\times G} dg_1 dg_2 d_T \phi(g_1,g_2) T(g_1,g_2)^\dagger = \sum_i p_i \left(\int_G dg_1 d_\pi \kappa_i(g_1)\pi(g_1)^\dagger \right) \otimes \left(\int_G dg_2 d_\tau \eta_i(g_2)\tau(g_2)^\dagger \right), \tag{14}$$

where $dg := dg_1 dg_2$ is the Haar measure on $G \times G$. Moreover, since we need to integrate in Eq. (14) in order to obtain a density matrix, it is enough that the decomposition (13) holds almost everywhere with respect to the measure dg. Hence we obtain the following theorem:

Theorem 1. Let G be a compact kinematical group; π, τ its irreducible representations. A state ϱ on $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$ is separable if and only if its noncommutative characteristic function ϕ_ϱ can be written as a convex combination: $\phi_\varrho(g_1,g_2) = \Sigma_i p_i \kappa_i(g_1)\eta_i(g_2)$, where $\kappa_i, \eta_i \in \mathcal{E}_1(G)$ and the equality holds almost everywhere with respect to the Haar measure on $G \times G$.

The above theorem is our group-theoretical reformulation of the separability problem. The generalization to arbitrary multipartite systems is straightforward. We call the functions possessing decompositions of the type (13) *separable* and otherwise—*entangled*. One may thus generalize the separability problem to groups in the following way:

Generalized separability problem. Given an arbitrary function $\phi \in \mathcal{P}_1(G\times G)$, decide whether it is separable or not.

This is an interesting mathematical problem, with connections to, e.g., properties of polynomials on groups: if $G=\mathrm{SU}(2)$, then, since ϕ_ϱ are polynomials in the group parameters, Theorem 1 states that a state is separable iff its group polynomial separates into two polynomials in the variables g_1 and g_2, respectively.

One of the potential advantages of the current approach is its universality. For example, for $G=\mathrm{SU}(2)$ characterization of separable functions within $\mathcal{P}_1(\mathrm{SU}(2)\times\mathrm{SU}(2))$ would lead through Eqs. (5) and (10) to the characterization of all separable states in all possible finite dimensions. The other, more conceptual, advantage will be discussed in Sec. VII. Note that if one considers a restriction $\phi|_{Abel}$ of an arbitrary $\phi \in \mathcal{P}_1(G\times G)$ to any Abelian subgroup of $G\times G$ (like a Cartan subgroup if G is a Lie group [29]), then the separable decomposition (13), possibly infinite, always exists. This follows from the fact that on Abelian groups the usual Fourier transform is available. For a concrete example consider $G=\mathrm{SU}(2)$. Then the maximal Abelian subgroup is $\mathrm{U}(1)\times\mathrm{U}(1)$ and one can always write

$$\phi(\theta_1,\theta_2)=\sum_{k,l}\hat{\phi}_{kl}e^{-ik\theta_1}e^{-il\theta_2}, \tag{15}$$

where the angles θ_1,θ_2 parametrize $\mathrm{U}(1)\times\mathrm{U}(1)$ and $\hat{\phi}_{kl}$ are the Fourier coefficients of $\phi|_{U(1)\times U(1)}$. Since $e^{-ik\theta}\in\mathcal{P}_1[\mathrm{U}(1)]$, $\hat{\phi}_{kl}\geqslant 0$ by Bochner's theorem [13], and $\Sigma_{kl}\hat{\phi}_{kl}=1$ by normalization of ϕ, the Fourier series (15) is just the separable decomposition of $\phi|_{U(1)\times U(1)}$. For characteristic functions of states, i.e., for $\phi=\phi_\varrho$, the series (15) is finite, $k=-2j_1,-2j_1+2,\dots,2j_1$, $l=-2j_2,-2j_2+2,\dots,2j_2$, where j_1, j_2 are the corresponding spins, as ϕ_ϱ's are polynomials of bi-degree $(2j_1,2j_2)$ in the group parameters; see the Appendix. However, for separable states the decomposition (15) will not generically prolong to the whole $\mathrm{SU}(2)\times\mathrm{SU}(2)$, as it contains at most $(2j_1+1)(2j_2+1)=mn$ terms, whereas from Caratheodory's Theorem we know that the number of terms in a separable decomposition is bounded by m^2n^2 [3]. We further develop the connection between group noncommutativity and entanglement in Sec. VII.

IV. ANALYSIS OF THE PPT CRITERION AND PURE STATES

In Sec. II we have seen that the language of noncommutative characteristic functions is as valid a description of quantum states as the usual language of density matrices. Hence, in particular, the known separability criteria should have their group-theoretical analogs. In this section we show how to translate the PPT criterion [1] and a simple criterion for pure states as two examples. Recall that the PPT condition implies that if ϱ is separable, then the partially transposed matrix ϱ^{T_1} is positive semidefinite [30].

Let us first note that for an arbitrary positive definite function ϕ it holds

$$\phi(g^{-1})=\overline{\phi(g)}, \tag{16}$$

and $\overline{\phi}$ is again positive definite. Hence, we immediately obtain from Eq. (13) a necessary separability criterion for an arbitrary $\phi\in\mathcal{P}_1(G\times G)$:

Proposition 1. If $\phi\in\mathcal{P}_1(G\times G)$ is separable then $\widetilde{\phi}(g_1,g_2):=\phi(g_1^{-1},g_2)\in\mathcal{P}_1(G\times G)$.

In particular, from Theorem 1 we obtain the implication: (ϱ-separable) $\Rightarrow\widetilde{\phi}_\varrho\in\mathcal{P}_1(G\times G)$. We will show that it is intimately related to the PPT condition. For that we will first consider $G=\mathrm{SU}(2)$:

Proposition 2. $\widetilde{\phi}_\varrho\in\mathcal{P}_1(\mathrm{SU}(2)\times\mathrm{SU}(2))$ if and only if $\varrho^{T_1}\geqslant 0$.

Proof. Let us first assume that $\varrho^{T_1}\geqslant 0$, so that $\phi_{\varrho^{T_1}}\in\mathcal{P}_1(\mathrm{SU}(2)\times\mathrm{SU}(2))$. The latter is just $\phi_{\varrho^{T_1}}(g_1,g_2)=\mathrm{tr}[\varrho\,\pi(g_1)^T\otimes\tau(g_2)]=\mathrm{tr}[\varrho\,\overline{\pi(g_1^{-1})}\otimes\tau(g_2)]$, and since for any $a\in\mathrm{SU}(2)$

$$\bar{a}=uau^{-1},\quad u:=-i\sigma_y, \tag{17}$$

and π polynomially depends on the group parameters (see Appendix), we obtain that $\phi_{\varrho^{T_1}}(g_1,g_2)=\phi_\varrho(ug_1^{-1}u^{-1},g_2)$. The condition (3) for $\phi_{\varrho^{T_1}}$ then takes the following form:

$$\begin{aligned}\int\int d\widetilde{g}d\widetilde{h}\overline{f(\widetilde{g})}\phi_{\varrho^{T_1}}(\widetilde{g}^{-1}\widetilde{h})f(\widetilde{h})\\ =\int dg_1dg_2\int dh_1dh_2\overline{f(g_1u,g_2)}\\ \times\widetilde{\phi}_\varrho(g_1^{-1}h_1,g_2^{-1}h_2)f(h_1u,h_2)\geqslant 0,\end{aligned} \tag{18}$$

where $\widetilde{g}:=(g_1,g_2)$. Since the inequality (18) is satisfied for any $f\in L^1(\mathrm{SU}(2)\times\mathrm{SU}(2))$, the right shift by u of the first argument is irrelevant. Thus, we get that $\widetilde{\phi}_\varrho\in\mathcal{P}_1(\mathrm{SU}(2)\times\mathrm{SU}(2))$ (the normalization follows trivially).

On the other hand, let us assume that $\widetilde{\phi}_\varrho\in\mathcal{P}_1(\mathrm{SU}(2)\times\mathrm{SU}(2))$. Then from the similar argument to that leading to the condition (9), we can construct a positive semidefinite operator,

$$\begin{aligned}\int_{G\times G}dg_1dg_2d_T\widetilde{\phi}_\varrho(g_1,g_2)T(g_1,g_2)^\dagger\\ =\int dg_1dg_2d_\pi d_\tau\phi_\varrho(g_1,g_2)\\ \times\overline{\pi(u)}[\pi(g_1)^\dagger]^T\pi(u)^T\otimes\tau(g_2)^\dagger\\ =[\overline{\pi(u)}\otimes\mathbf{1}]\varrho^{T_1}[\pi(u)^T\otimes\mathbf{1}]\geqslant 0,\end{aligned} \tag{19}$$

where in the first step we used the fact that $dg^{-1}=dg$. Since the local unitary rotation by $\pi(u)^T\otimes\mathbf{1}$ does not affect the positivity of the operator in the inequality (19), the latter is equivalent to $\varrho^{T_1}\geqslant 0$. ■

The crucial role in the above proof, especially in obtaining the inequality (19), has been played by the relation (17), implying a unitary equivalence, denoted by $\sim$, between $\mathrm{SU}(2)$-representations τ_k and their complex conjugates $\overline{\tau_k}$ for all k: $\overline{\tau_k}=C_k\tau_kC_k^\dagger$, i.e., $\tau_k\sim\overline{\tau_k}$. The intertwining isomorphisms C_k, equal to $\tau_k(u)$ for this particular group, satisfy $\overline{C_k}C_k=\mathbf{1}$.

Representations with such properties are called representations of real type [31].

Now a natural question arises: if we consider a kinematical group which possesses at least one irreducible representation $\pi \nsim \overline{\pi}$ (for example $G=\mathrm{SU}(3)$, $\pi=\mathrm{id}$), can we obtain from Proposition 1 any new criterion, independent from the PPT condition? The negative answer provides the next theorem.

Theorem 2. Let G be a compact kinematical group; π, τ its irreducible representations. For any state ϱ on $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$, $\varrho^{T_1} \geqslant 0$ if and only if $\widetilde{\phi}_\varrho \in \mathcal{P}_1(G\times G)$.

Proof. For a general group G the property (17) does not hold and we cannot use the previous technique. However, $\widetilde{\phi}_\varrho$ can be represented as follows:

$$\widetilde{\phi}_\varrho(g_1,g_2) = \mathrm{tr}[\varrho^{T_1}\overline{\pi(g_1)} \otimes \tau(g_2)], \tag{20}$$

so that $\widetilde{\phi}_\varrho$ becomes a noncommutative characteristic function of ϱ^{T_1}, treated as an operator acting on $\mathcal{H}_{\overline{\pi}}\otimes\mathcal{H}_\tau$. Since $\overline{\pi}$ is irreducible iff π is, we can invert the transformation (20),

$$\varrho^{T_1} = \int_{G\times G} dg_1 dg_2 d_\pi d_\tau \widetilde{\phi}_\varrho(g_1,g_2)\overline{\pi}(g_1)^\dagger \otimes \tau(g_2)^\dagger. \tag{21}$$

Then the statement follows immediately from the general results of Sec. II: if $\widetilde{\phi}_\varrho \in \mathcal{P}_1(G\times G)$, positivity of ϱ^{T_1} follows from the same argument as that leading to the inequality (9). On the other hand, if $\varrho^{T_1}\geqslant 0$ then a direct calculation shows that $\widetilde{\phi}_\varrho$ satisfies the condition (3). ■

Let us now briefly examine pure states. For pure states a number of necessary and sufficient separability conditions is available. The one which is most easily translated into the group-theoretical language is the following:

$$\psi \in \mathcal{H} \textit{ is product } \Leftrightarrow \mathrm{tr}_1(\mathrm{tr}_2|\psi\rangle\langle\psi|)^2 = 1 = \mathrm{tr}_2(\mathrm{tr}_1|\psi\rangle\langle\psi|)^2.$$

Using the orthogonality of matrix elements of representations, we easily obtain that this criterion is equivalent to the following integral condition:

Proposition 3. A function $\phi \in \mathcal{E}_1(G\times G)$ is a product if and only if:

$$\int_G dg_1 d_\pi|\phi(g_1,e)|^2 = 1 = \int_G dg_2 d_\tau|\phi(e,g_2)|^2. \tag{22}$$

Note that the above condition applies to an *arbitrary* $\phi \in \mathcal{E}_1(G\times G)$ since, as we mentioned in Sec. II, every $\phi \in \mathcal{E}_1(G\times G)$ is of the form ϕ_ψ for some irreducible representation $\pi\otimes\tau$ of $G\times G$ and some pure state $\psi\in\mathcal{H}_\pi\otimes\mathcal{H}_\tau$.

V. ANALYSIS ON FINITE GROUPS

In this section we study the special case of finite kinematical groups. An example of such groups are symmetric groups $\mathfrak{S}_M$ (group of permutations of M elements) and moreover, every finite group is isomorphic to a subgroup of some $\mathfrak{S}_M$ [29]. Finite groups are in particular compact, and hence all of the previous theory applies to them as well, with the only change being,

$$\int_G dg \to \frac{1}{|G|}\sum_{g\in G}, \tag{23}$$

where $|G|$ is the number of elements of G (its order). However, for finite groups several simplifications occur. First of all, if $|G|=N$, then the space of complex functions on G is isomorphic to $\mathbb{C}^N$, and we may identify each function ϕ with a row vector $\vec{\phi}$ of its values. The positive definiteness condition (3) then takes the following form:

$$\sum_{\alpha,\beta=1}^{N} \overline{c_\alpha}\phi(g_\alpha^{-1}g_\beta)c_\beta \geqslant 0 \quad \text{for any } \vec{c}\in\mathbb{C}^N, \tag{24}$$

(indices $\alpha,\beta,\ldots$ now enumerate the group elements), which is just the positive semidefiniteness condition for the matrix

$$\mathbf{\Phi}_{\alpha\beta} := \phi(g_\alpha^{-1}g_\beta) \tag{25}$$

(compare with [22]). To examine more closely the structure of this matrix, let us first fix the labeling of the group elements such that $g_1:=e$. Then, the first row of $\mathbf{\Phi}$ contains the values of the function ϕ itself, and hence, it determines the rest of the matrix. We may define a function σ on $\mathbb{N}\times\mathbb{N}$ through:

$$g_{\sigma(\alpha,\beta)} := g_\alpha^{-1}g_\beta. \tag{26}$$

Note that σ is completely determined by the group multiplication table, $\sigma(\alpha,\alpha)=1$, and it satisfies the cocycle condition

$$g_{\sigma(\alpha,\beta)}g_{\sigma(\beta,\gamma)} = g_{\sigma(\alpha,\gamma)} \tag{27}$$

(no summation over β here). Combining Eq. (26) with the normalization condition (4), and the property (16), we obtain a general form of the matrix (25) for an arbitrary $\phi \in \mathcal{P}_1(G)$

$$\mathbf{\Phi} = \begin{bmatrix} 1 & \phi_2 & \phi_3 & \cdots & \phi_N \\ \overline{\phi_2} & 1 & \phi_{\sigma(2,3)} & \cdots & \phi_{\sigma(2,N)} \\ \overline{\phi_3} & \overline{\phi_{\sigma(2,3)}} & 1 & \cdots & \phi_{\sigma(3,N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\phi_N} & \overline{\phi_{\sigma(2,N)}} & \overline{\phi_{\sigma(3,N)}} & \cdots & 1 \end{bmatrix}. \tag{28}$$

In other words, the matrix $\mathbf{\Phi}$ is built from the vector $\vec{\phi}$ by permuting in each row (or column) its components according to the multiplication table of G. Relabeling of the group elements corresponds to unitary rotation of $\mathbf{\Phi}$, which does not affect the condition (24), and hence we may work with a fixed labeling.

For pure states, one can rewrite the condition (7) in the following form:

$$\phi_\varrho(g^{-1}g') = \int_G dh d_\tau \phi_\varrho(g^{-1}h)\phi_\varrho(h^{-1}g'), \tag{29}$$

from which it follows that ϱ is pure if and only if

$$\mathbf{\Phi}_\varrho^2 = \frac{N}{d_\tau}\mathbf{\Phi}_\varrho. \tag{30}$$

Hence, $(d_\tau/N)\mathbf{\Phi}_\varrho$ is a projector.

Let us now move to bipartite systems, i.e., to systems with the kinematical group $G\times G$. We may view functions ϕ on such group either as $N\times N$ matrices $\phi_{\alpha\beta}:=\phi(g_\alpha,g_\beta)$, or as vectors from $\mathbb{C}^{N^2}$. The separability criterion—Theorem 1—then takes the following form on finite G.

Proposition 4. A function $\phi\in\mathcal{P}_1(G\times G)$ is separable if and only if there exists a convex decomposition:

$$\phi_{\alpha\beta}=\sum_i p_i\kappa_{i\alpha}\eta_{i\beta}, \tag{31}$$

where for each i vectors $\vec{\kappa}_i,\vec{\eta}_i\in\mathbb{C}^N$ lead, according to the prescription (28), to positive semidefinite matrices.

Decomposition (31) resembles the singular value decomposition of the matrix $\phi_{\alpha\beta}$, however, the vectors are specifically constrained. Let us mention another equivalent form of Proposition 4.

Proposition 5. A function $\phi\in\mathcal{P}_1(G\times G)$ is separable if and only if its matrix $\mathbf{\Phi}$ defined by Eq. (25) can be convexly decomposed as follows:

$$\mathbf{\Phi}=\sum_i p_i\boldsymbol{K}_i\otimes\boldsymbol{N}_i, \tag{32}$$

where for each i, $\boldsymbol{K}_i,\boldsymbol{N}_i\geqslant 0$ and are of the form (28) for some $\vec{\kappa}_i,\vec{\eta}_i\in\mathbb{C}^N$.

The proof follows for the fact that the first row of the matrix equality (32) is just the Eq. (31), and from the specific structure (28) of the matrices in Eq. (32).

From the condition (24), the matrix $\mathbf{\Phi}_{\alpha\alpha',\beta\beta'}=\phi(g_\alpha^{-1}g_\beta,g_{\alpha'}^{-1}g_{\beta'})$ is positive semidefinite as an operator on $\mathbb{C}^N\otimes\mathbb{C}^N$, and, after rescaling by $1/N^2$, has trace one. Hence, Proposition 5 embeds the given separability problem into the higher dimensional one [32]. Note, however, that the matrices in Eq. (32) are of a very specific form: they are completely determined by their first rows and the group multiplication table. The necessary separability criterion from Proposition 1 takes a particularly familiar form for finite groups:

Proposition 6. If $\phi\in\mathcal{P}_1(G\times G)$ is separable then $\mathbf{\Phi}^{T_1}\geqslant 0$.

The proof follows from Proposition 1, the equality: $\widetilde{\mathbf{\Phi}}_{\alpha\alpha',\beta\beta'}=\widetilde{\phi}(g_\alpha^{-1}g_\beta,g_{\alpha'}^{-1}g_{\beta'})=\phi(g_\beta^{-1}g_\alpha,g_{\alpha'}^{-1}g_{\beta'})=\mathbf{\Phi}_{\beta\alpha',\alpha\beta'}$, and the positive definiteness condition (24).

VI. FORMAL RESEMBLANCE TO LOCAL HIDDEN VARIABLE MODELS

Let us here remark on a purely formal resemblance of the group-theoretical formalism from the preceding sections to LHV models [18]. Following the usual approach, let us consider an expectation value of a product operator $A\otimes B$, where $A=\Sigma_\mu a_\mu P_\mu$, $B=\Sigma_\nu b_\nu Q_\nu$ are the corresponding spectral decompositions. Using the representation (5), the mean value of $A\otimes B$ in the state ϱ can be written as follows:

$$\mathrm{tr}(A\otimes B\varrho)=\sum_{\mu,\nu}a_\mu b_\nu\int_{G\times G}dg_1dg_2d_\pi d_\tau\phi_\varrho(g_1,g_2)\times\mathrm{tr}[P_\mu\pi(g_1)^\dagger]\mathrm{tr}[Q_\nu\tau(g_2)^\dagger]. \tag{33}$$

Hence, the probability $p(\mu,\nu|A,B)$ of obtaining the value a_μ for A and b_ν for B is given by

$$p(\mu,\nu|A,B)=\int dg_1dg_2d_\pi d_\tau\phi_\varrho(g_1,g_2)\times\mathrm{tr}[P_\mu\pi(g_1)^\dagger]\mathrm{tr}[Q_\nu\tau(g_2)^\dagger]. \tag{34}$$

This expression *formally* resembles a LHV model, where the role of the probability space plays $G\times G$, the "response functions" are $R(\mu,g_1):=\mathrm{tr}[P_\mu\pi(g)^\dagger]$ and $R(\nu,g_2):=\mathrm{tr}[Q_\nu\tau(g_2)^\dagger]$, and the "probability measure" is $dm:=d_\pi d_\tau\phi_\varrho(g_1,g_2)dg_1dg_2$. The resemblance is of course only formal, since the response functions, as well as the measure dm, are complex. The response functions satisfy only $R(g^{-1})=\overline{R(g)}$, while the measure dm is positive definite but not positive.

VII. NONCOMMUTATIVITY AND ENTANGLEMENT

We conclude with a general remark, connecting the existence of entanglement with noncommutativity of the kinematical group G. For that we first have to change the usual mathematical language of quantum statistics (we do not consider dynamics here). Instead of using Hilbert spaces and density matrices, let us (i) assume that the kinematical arena is set up by the kinematical group G; (ii) represent physical states by functions from $\mathcal{P}_1(G)$ (or its subset) rather than by density matrices; (iii) for composite systems, take as the kinematical group the product group $G\times G\times G\ldots$ (for alternative group-theoretical reformulations see, e.g., Refs. [33–35]). As we have seen in Sec. II, such a description is indeed equivalent to the standard one, provided that the kinematical group is chosen correctly: for spin systems $G=\mathrm{SU}(2)$, for canonically quantized particles it is the Heisenberg-Weyl group [19], while for classical particles G is just the phase-space $\mathbb{R}^{2n}$.

Now, let us assume that the kinematical group is Abelian. Then by Bochner's theorem [13], our states, i.e., functions from $\mathcal{P}_1(G)$, are in one-to-one correspondence with Borel probability measures on the space of all irreducible representations of G, $\hat{G}$, which in this case is also an Abelian group (for instance $\widehat{\mathbb{R}^{2n}}\simeq\mathbb{R}^{2n}$). Hence, we recover classical statistical description of our system [21], with $\hat{G}$ playing the role of the phase-space (at least for the purpose of statistics). If, moreover, the system under consideration is multipartite, then due to the fact that $\widehat{G_1\times G_2}=\hat{G}_1\times\hat{G}_2$ [13], the phase-space of the composite system is the usual Cartesian product of the individual phase-spaces, and our states correspond to the probability measures on this product. There is no place for entanglement here, understood as the impossibility of generating the composite system state space from the individual state spaces, because probability measures on Cartesian products can always be decomposed (under suitable limits) into the convex mixtures of product measures (due to the underlying structure of the σ-algebra of Borel sets).

On the other hand, when G is non-Abelian, then Bochner's theorem cannot be applied, and $\mathcal{P}_1(G)$ is in one-to-one correspondence with density matrices through the inverse Fourier transform (5) and (10). Since density matrices exhibit entanglement, one may view the latter as the consequence of the noncommutativity of the kinematical group G. The last observation opens some possibility of speculations on the connection between entanglement and the uncertainty principles. In this context we note that Gühne has developed in Ref. [7] some methods of entanglement description with the help of uncertainty relations.

Let us also mention that the general group-theoretical approach, sketched above, can be also applied to canonically quantized systems and the analysis of the correspondence principle [19].

Finally, as discussed at the end of the Appendix, our approach opens a possibility of deriving highly nontrivial statements on positive definite functions on product groups, using the theory of entanglement.

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APPENDIX: SU(2)-CHARACTERISTIC FUNCTION OF THE 3⊗3 HORODECKI'S STATE

As an example we calculate for $G=\mathrm{SU}(2)$ the characteristic function of the $3\otimes 3$ PPT entangled state, discovered by Horodecki in Ref. [3]. Since any irreducible representation T of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is of the form $T=\tau_{j_1}\otimes\tau_{j_2}$ for some spins j_1, j_2, all we need are the matrix elements $\tau^j_{\mu\nu}$ of the corresponding spin-j representations τ_j of SU(2). The concrete basis $\{e_\mu\}$ in which we calculate them is irrelevant for our purposes, as from Eq. (2) it follows that a change of basis: $\tau_{j_1}\mapsto \mathrm{U}_1\tau_{j_1}\mathrm{U}_1^\dagger$, $\tau_{j_2}\mapsto \mathrm{U}_2\tau_{j_2}\mathrm{U}_2^\dagger$ induces only a local rotation of the state ϱ:

$$\mathrm{tr}[\varrho \mathrm{U}_1\tau_{j_1}\mathrm{U}_1^\dagger\otimes \mathrm{U}_2\tau_{j_2}\mathrm{U}_2^\dagger]=\mathrm{tr}[(\mathrm{U}_1^\dagger\otimes \mathrm{U}_2^\dagger\varrho \mathrm{U}_1\otimes \mathrm{U}_2)\tau_{j_1}\otimes\tau_{j_2}], \tag{A1}$$

and the rotated state $\mathrm{U}_1^\dagger\otimes \mathrm{U}_2^\dagger\varrho \mathrm{U}_1\otimes \mathrm{U}_2$ is separable if and only if ϱ is separable. The above remark concerning bases obviously applies to any kinematical group G.

A convenient formula for $\tau^j_{\mu\nu}$ can be found, for example, in Ref. [29],

$$\tau^j_{\mu\nu}(g)=\frac{1}{(j-\mu)!}\left.\frac{d^{j-\mu}}{dz^{j-\mu}}\right|_0[(\alpha z+\beta)^{j-\nu}(-\bar{\beta}z+\bar{\alpha})^{j+\nu}], \tag{A2}$$

where $\mu,\nu=-j,-j+1,\ldots,j$ and α, β are the group parameters

$$g=\begin{bmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{bmatrix},\quad |\alpha|^2+|\beta|^2=1. \tag{A3}$$

From Eq. (A2) we immediately see that matrix elements of the representation τ_j are homogeneous polynomials of degree $2j$ in the group parameters. Hence, matrix elements of $\tau_{j_1}\otimes\tau_{j_2}$ are polynomials of bi-degree $(2j_1,2j_2)$ in (α_1,β_1), (α_2,β_2) as mentioned in Sec. II.

The $3\otimes 3$ Horodecki's state is given by

$$\varrho=\frac{1}{8a+1}\begin{bmatrix} a&0&0&0&a&0&0&0&a\\ 0&a&0&0&0&0&0&0&0\\ 0&0&a&0&0&0&0&0&0\\ 0&0&0&a&0&0&0&0&0\\ a&0&0&0&a&0&0&0&a\\ 0&0&0&0&0&a&0&0&0\\ 0&0&0&0&0&0&\frac{1+a}{2}&0&\frac{\sqrt{1-a^2}}{2}\\ 0&0&0&0&0&0&0&a&0\\ a&0&0&0&a&0&\frac{\sqrt{1-a^2}}{2}&0&\frac{1+a}{2}\end{bmatrix}, \tag{A4}$$

where $0\leqslant a\leqslant 1$. From Eq. (A2) we find the three dimensional representation of SU(2)

$$\tau_1(g)=\begin{bmatrix}\alpha^2 & -\alpha\bar{\beta} & \bar{\beta}^2\\ 2\alpha\beta & |\alpha|^2-|\beta|^2 & -2\overline{\alpha\beta}\\ \beta^2 & \bar{\alpha}\beta & \bar{\alpha}^2\end{bmatrix}. \tag{A5}$$

Inserting Eqs. (A4) and (A5) into Eq. (2), we obtain the characteristic function of the state (A4)

$$\begin{aligned}\phi_\varrho(g_1,g_2)=\frac{a}{8a+1}\Bigg[&\left(\alpha_1^2+\frac{1}{2}\overline{\alpha_1}^2\right)(\alpha_2^2+\overline{\alpha_2}^2)+(\beta_1\beta_2)^2\\ &+(\overline{\beta_1\beta_2})^2+4\alpha_1\beta_1\alpha_2\beta_2+4\overline{\alpha_1\beta_1\alpha_2\beta_2}\\ &+\alpha_1\overline{\beta_1}\alpha_2\overline{\beta_2}+\overline{\alpha_1}\beta_1\overline{\alpha_2}\beta_2+(\alpha_1^2+\overline{\alpha_1}^2)(|\alpha_2|^2\\ &-|\beta_2|^2)+(|\alpha_1|^2-|\beta_1|^2)(\alpha_2^2+\overline{\alpha_2}^2)\\ &+(|\alpha_1|^2-|\beta_1|^2)(|\alpha_2|^2-|\beta_2|^2)\Bigg]\\ &+\frac{\sqrt{1-a^2}}{2}\overline{\alpha_1}^2(\beta_2^2+\overline{\beta_2}^2)+\frac{1}{2}\overline{\alpha_1}^2(\alpha_2^2+\overline{\alpha_2}^2).\end{aligned} \tag{A6}$$

Note that from the fact that the state (A4) is entangled for $0<a<1$, we obtain through Theorem 1, a highly nontrivial result concerning the function (A6): the function (A6) cannot be represented as a convex mixture of products of positive definite functions, depending on parameters (α_1,β_1), and (α_2,β_2), respectively.

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